CONTENTS

MATHEMATICS - XI

SETS

1.1 SETS

It is a well known fact that any attempt to define a set has always led mathematicians to unsurmountable difficulties. For example, suppose one defines the term set as "*a well defined collection of objects*". One may then ask what is meant by a collection. If one answers that a collection is an aggregate of objects or things. What is then an aggregate? Perhaps then one may define that an aggregate is a class of things. What is then a class? Now one may define a class as a collection. In this manner question after question, since our language is finite, we find that after some time we will have to use some words which have already been questioned. The definition thus becomes circular and worthless. Thus mathematicians realized that there must be some undefined (or primitive) terms. In this chapter we start with two undefined (or primitive) terms – "element" and "set". We assume that the word "set" is synonymous with the words "collection", "aggregate", "class" and is comprised of elements. The words "element", "object", "member" are synonymous.

If a is an element of a set A, then we write $a \in A$ and say a belongs to A or a is in A or a is a member of A. If a does not belong to A, the we write $a \notin A$. It is assumed here that if A is any set and a is any element, then either $a \in A$ or $a \notin A$ and the two possibilities are mutually exclusive. Thus one cannot say "consider the set A of some positive integers", because it is not sure whether $3 \in A$ or $3 \notin A$.

Throughout this chapter we shall denote sets by capital alphabets e.g. A, B, C, X, Y, Z etc. and the elements by the small alphabets e.g. a, b, c, x, y, z etc.

The following are some illustrations of sets:

ILLUSTRATION 1 *The collection of vowels in English alphabets. This set contains five elements, namely, a, e, i, o, u.*

ILLUSTRATION 2 *The collection of first five prime natural numbers is a set containing the elements 2, 3, 5, 7, 11.*

ILLUSTRATION 3 *The collection of all states in the Indian union is a set.*

ILLUSTRATION 4 *The collection of past presidents of the Indian union is a set.*

ILLUSTRATION 5 *The collection of cricketers in the world who were out for 99 runs in a test match is a set.*

ILLUSTRATION 6 *The collection of good cricket players of India is not a set, since the term "good player is vague and it is not well defined".*

Similarly, collection of good teachers in a school is not a set. However, the collection of all teachers in a school is a set.

In this chapter we will have frequent interaction with some sets, so we reserve some letters for these sets as listed below:

N : for the set of natural numbers.

Z : for the set of integers.
Z^+ : for the set of all positive integers.
Q : for the set of all rational numbers.
Q^+ : for the set of all positive rational numbers.
R : for the set of all real numbers.
R^+ : for the set of all positive real numbers.
C : for the set of all complex numbers.

EXERCISE 1.1

1. What is the difference between a collection and a set? Give reasons to support your answer?
2. Which of the following collections are sets? Justify your answer:
   (i) A collection of all natural numbers less than 50.
   (ii) The collection of good hockey players in India.
   (iii) The collection of all girls in your class.
   (iv) The collection of most talented writers of India. [NCERT]
   (v) The collection of difficult topics in mathematics.
   (vi) The collection of all months of a year beginning with the letter J. [NCERT]
   (vii) A collection of novels written by Munshi Prem Chand. [NCERT]
   (viii) The collection of all questions in this chapter. [NCERT]
   (ix) A collection of most dangerous animals of the world. [NCERT]
   (x) The collection of prime integers.
3. If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then insert the appropriate symbol $\in$ or $\notin$ in each of the following blank spaces:
   (i) $4 \_\_ A$ (ii) $-4 \_\_ A$ (iii) $12 \_\_ A$
   (iv) $9 \_\_ A$ (v) $0 \_\_ A$ (vi) $-2 \_\_ A$

ANSWERS

1. Every set is a collection but a collection is not necessarily a set. Only well defined collections are sets. For example, collection of good cricket players is a collection but it is not a set.
2. (i), (iii), (vi), (vii), (viii), (x)
3. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (iv) $\in$ (v) $\in$ (vi) $\notin$

HINTS TO NCERT & SELECTED PROBLEMS

2. (iv) The collection of most talented writers of India is not a set as there is no specific criterion to determine whether a writer is talented or not.
   (vi) The collection of all months of a year beginning with the letter J is a set given by {January, June, July}.
   (vii) The collection of novels written by Munshi Prem Chand is a set because one can determine whether a novel is written by him or not.
   (viii) The collection of all questions in this chapter is a set because if a question is given one can easily decide whether it is a question of this chapter or not.
   (ix) The collection of most dangerous animals of the world is not a set because there is no criterion to determine whether an animal is most dangerous or not.

1.2 DESCRIPTION OF A SET

A set is often described in the following two forms. One can make use of any one of these two ways according to his (her) convenience.
(i) Roster form or Tabular form
(ii) Set-builder form

1.2.1 ROSTER FORM In this form a set is described by listing elements, separated by commas, within braces { }.

ILLUSTRATION 1 *The set of vowels of English Alphabet may be described as* $\{a, e, i, o, u\}$.

ILLUSTRATION 2 *The set of even natural numbers can be described as* $\{2, 4, 6, ...\}$. *Here the dots stand for 'and so on'.*

ILLUSTRATION 3 *If A is the set of all prime numbers less than 11, then* $A = \{2, 3, 5, 7\}$.

NOTE *The order in which the elements are written in a set makes no difference. Thus,* $\{a, e, i, o, u\}$ *and* $\{e, a, i, o, u\}$ *denote the same set. Also, the repetition of an element has no effect. For example,* $\{1, 2, 3, 2\}$ *is the same set as* $\{1, 2, 3\}$.

1.2.2 SET-BUILDER FORM In this form, a set is described by a characterizing property $P(x)$ of its elements x. In such a case the set is described by $\{x : P(x) \text{ holds}\}$ or, $\{x \mid P(x) \text{ holds}\}$, which is read as 'the set of all x such that $P(x)$ holds'. The symbol '|' or ':' is read as 'such that'.

In other words, in order to describe a set, a variable x (say) (to denote each element of the set) is written inside the braces and then after putting a colon the common property $P(x)$ possessed by each element of the set is written within the braces.

ILLUSTRATION 4 *The set E of all even natural numbers can be written as*

$$E = \{x : x \text{ is a natural number and } x = 2n \text{ for } n \in N\}$$

or, $E = \{x : x \in N, x = 2n, n \in N\}$ or, $E = \{x \in N : x = 2n, n \in N\}$

ILLUSTRATION 5 *The set* $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *can be written as* $A = \{x \in N : x \leq 8\}$.

ILLUSTRATION 6 *The set of all real numbers greater than –1 and less than 1 can be described as* $\{x \in R : -1 < x < 1\}$.

ILLUSTRATION 7 *The set* $A = \{0, 1, 4, 9, 16, ...\}$ *can be written as* $A = \{x^2 : x \in Z\}$.

ILLUSTRATIVE EXAMPLES

Type I ON DESCRIBING OR REPRESENTING SETS IN TABULAR FORM OR ROSTER FORM

EXAMPLE 1 *Describe the following sets in Roster form:*
(i) *The set of all letters in the word 'MATHEMATICS'*
(ii) *The set of all letters in the word 'ALGEBRA'*
(iii) *The set of all vowels in the word 'EQUATION'*
(iv) *The set of all natural numbers less than 7.*
(v) *The set of squares of integers.*

SOLUTION (i) We observe that distinct letters in the word 'MATHEMATICS' are
M, A, T, H, E, I, C, S

Since the order in which the elements of a set are written is immaterial and the repetition of elements has no effect. So, required set can be described as follows:
$\{M, A, T, H, E, I, C, S\}$

(ii) We find that the word 'ALGEBRA' has following distinct letters:
A, L, G, E, B, R

Hence, required set can be described in Roster form as follows:

$\{A, L, G, E, B, R\}$

(iii) Clearly, word 'EQUATION' has following vowels:

A, E, I, O, U

So, required set can be described as follows:

$\{A, E, I, O, U\}$

(iv) Natural numbers less than 7 are: 1, 2, 3, 4, 5, 6.

Hence, required set can be described as follows:

$\{1, 2, 3, 4, 5, 6\}$

(v) Since square of a negative integer is same as the square of its absolute value. Therefore, squares of integers are 0, 1, 4, 9, 16, 25, ...

Hence, required set is $\{0, 1, 4, 9, 16, \ldots\}$

Type II ON DESCRIBING OR REPRESENTING SETS IN SET-BUILDER FORM

EXAMPLE 2 *Describe the following sets in set-builder form:*

(i) *The set of all letters in the word 'PROBABILITY'.*

(ii) *The set of reciprocals of natural numbers.*

(iii) *The set of all odd natural numbers.*

(iv) *The set of all even natural numbers.*

SOLUTION (i) Given set in set-builder form can be described as follows:

$\{x : x$ is a letter in the word 'PROBABILITY'$\}$

(ii) Given set can be described in set-builder form as follows:

$\{x : x$ is reciprocal of a natural number$\}$

or

$\left\{x : x = \frac{1}{n}, n \in N\right\}$ or, $\left\{\frac{1}{n} : n \in N\right\}$

(iii) An odd natural number can be written in the form $(2n - 1)$. So, given set can be described as follows:

$\{x : x = 2n - 1, n \in N\}$ or, $\{2n - 1 : n \in N\}$

(iv) An even natural number can be written as $2n$, where $n \in N$. Therefore, set of all even natural numbers can be written in the form

$\{x : x = 2n, n \in N\}$ or, $\{2n : n \in N\}$

EXAMPLE 3 *Write the set of all integers whose cube is an even integer.*

SOLUTION We know that the cube of an even integer is also an even integer. Hence, the required set is the set of all even integers which can also be written in the set-builder form as

$\{2n : n \in Z\}$.

EXAMPLE 4 *Write the set of all real numbers which cannot be written as the quotient of two integers in the set-builder form.*

SOLUTION We know that all rational numbers are expressible as the quotient of two integers. Therefore, the required set is the set of all irrational numbers which can be written as

$\{x : x$ is real and irrational$\}$ or, $\{x : x \in R$ but $x \notin Q\}$.

Type III ON DESCRIBING A SET IN ROSTER FORM WHEN IT IS GIVEN IN SET-BUILDER FORM

EXAMPLE 5 *Describe each of the following sets in Roster form:*

(i) $\{x : x$ *is a positive integer and a divisor of 9*$\}$

(ii) $\{x : x \in Z$ *and* $|x| \le 2\}$

(iii) $\{x : x$ *is a letter of the word 'PROPORTION'*$\}$

(iv) $\left\{x : \frac{n}{n^2+1}\right.$ *and* $\left. 1 \le n \le 3, \text{ where } n \in N\right\}$

SOLUTION (i) Since x is a positive integer and a divisor of 9.

So, x can take values 1, 3, 9.

$\therefore$ $\{x : x$ is a positive integer and a divisor of 9$\} = \{1, 3, 9\}$

(ii) We find that x is an integer satisfying $|x| \le 2$.

$\therefore$ $|x| = 0, 1, 2 \Rightarrow x = 0, \pm 1, \pm 2$

So, x can take values $-2, -1, 0, 1, 2$

$\therefore$ $\{x : x \in Z$ and $|x| \le 2\} = \{-2, -1, 0, 1, 2\}$

(iii) We find that distinct letters in the word 'PROPORTION' are P, R, O, T, N, I

So, x can be P, R, O, T, I, N

Hence, $\{x : x$ is a letter in the word 'PROPORTION'$\} = \{P, R, O, T, I, N\}$

(iv) We have,

$$x = \frac{n}{n^2+1} \text{ where } n \in N \text{ and } 1 \le n \le 3.$$

$$\therefore \quad n = \frac{n}{n^2+1}, \text{ where } n = 1, 2, 3$$

$$\Rightarrow \quad x = \frac{n}{1^2+1}, \frac{2}{2^2+1}, \frac{3}{3^2+1}$$

$$\Rightarrow \quad x = \frac{1}{2}, \frac{2}{5}, \frac{3}{10}$$

Hence, $\left\{x : x = \frac{n}{n^2+1} \text{ and } 1 \le n \le 3, \text{ where } n \in N\right\} = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}\right\}$

EXAMPLE 6 *Write the set of all vowels in English alphabet which precede s.*

SOLUTION The vowels in English alphabet which precede s are a, e, i, o. So, the set $A = \{a, e, i, o\}$ is the set of all vowels in English aplhabet which precedes.

EXAMPLE 7 *Write the set* $A = \{x \mid x \in Z, x^2 < 20\}$ *in the roster form.*

SOLUTION We observe that the squares of integers $0, \pm 1, \pm 2, \pm 3, \pm 4$ are less than 20. Therefore, the set A in roster form is

$A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

EXAMPLE 8 *Match each of the set on the left described in the roster form with the same set on the right described in the set-builder form:* **[NCERT]**

(i) $\{P, R, I, N, C, A, L\}$	(a) $\{x : x$ *is a positive integer and is a divisor of 18*$\}$
(ii) $\{0\}$	(b) $\{x : x$ *is an integer and* $x^2 - 9 = 0\}$
(iii) $\{1, 2, 3, 6, 9, 18\}$	(c) $\{x : x$ *is an integer and* $x + 1 = 1\}$
(iv) $\{-3, 3\}$	(d) $\{x : x$ *is a letter of the word 'PRINCIPAL'*$\}$

SOLUTION We have,

(i) $\{P, R, I, N, C, A, L\}$
$= \{P, R, I, N, C, I, P, A, L\}$
$= \{x : x$ is a letter of the word 'PRINCIPAL'$\}$
Hence, (i) matches with (d).

(ii) $\{0\} = \{x : x$ is an integer equal to zero$\}$
$= \{x : x$ is an integer and $x + 1 = 1\}$
Hence, (ii) matches with (c).

(iii) $\{1, 2, 3, 6, 9, 18\}$ = Set of all positive divisors of 18
$= \{x : x$ is a positive integer and is a divisor of $18\}$
Hence, (iii) matches with (a).

(iv) $\{-3, 3\} = \{x : x$ is an integer and $x^2 - 9 = 0\}$
Hence, (iv) matches with (b).

Type IV ON DESCRIBING A SET IN ROSTER FORM WHEN IT IS GIVEN SET-BUILDER FORM

EXAMPLE 4 Write the set $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}\right\}$ in the set-builder form. [NCERT]

SOLUTION We observe that each element in the given set has the denominator one more than the numerator. Also, the numerator begins from 1 and do not exceed 9. Hence, in the set-builder form the given set can be written as

$$\left\{x : x = \frac{n}{n+1}, n \in N, n \le 9\right\}$$

EXAMPLE 5 Write the set $X = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right\}$ in the set-builder form.

SOLUTION We observe that the elements of set X are the reciprocals of the squares of all natural numbers. So, the set X in set-builder form is

$$X = \left\{\frac{1}{n^2} : n \in N\right\}$$

EXAMPLE 6 Write the following sets in Roster form:

(i) $A = \{a_n : n \in N, a_{n+1} = 3a_n$ and $a_1 = 1\}$
(ii) $B = \{a_n : n \in N, a_{n+2} = a_{n+1} + a_n, a_1 = a_2 = 1\}$

SOLUTION (i) We have,
$a_1 = 1$ and $a_{n+1} = 3a_n$ for all $n \in N$
Putting $n = 1$ in $a_{n+1} = 3a_n$, we get
$a_2 = 3a_1 = 3 \times 1 = 3$ $[\because a_1 = 1]$
Putting $n = 2$ in $a_{n+1} = 3a_n$, we get
$a_3 = 3a_2 = 3 \times 3 = 3^2$ $[\because a_2 = 3]$
Putting $n = 3$ in $a_{n+1} = 3a_n$, we get
$a_4 = 3a_3 = 3 \times 3^2 = 3^3$ $[\because a_3 = 3^2]$
Similarly, we have
$a_5 = 3a_4 = 3 \times 3^3 = 3^4$

$a_6 = 3a_5 = 3 \times 3^4 = 3^5$ and so on.

Hence, $A = \{a_1, a_2, a_3, a_4, a_5, a_6, \ldots\} = \{1, 3, 3^2, 3^3, 3^4, 3^5, \ldots\}$

(ii) We have,
$a_1 = 1, a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$.
Putting $n = 1, 2, 3, 4, \ldots$ in $a_{n+2} = a_{n+1} + a_n$, we get
$a_3 = a_2 + a_1 = 1 + 1 = 2$
$a_4 = a_3 + a_2 = 2 + 1 = 3$
$a_5 = a_4 + a_3 = 3 + 2 = 5$
$a_6 = a_5 + a_4 = 5 + 3 = 8$ and so on.
Hence, $B = \{a_1, a_2, a_3, a_4, a_5, a_6, \ldots\} = \{1, 1, 2, 3, 5, 8, \ldots\}$

EXERCISE 1.2

1. Describe the following sets in Roster form:
   (i) $\{x : x$ is a letter before e in the English alphabet$\}$
   (ii) $\{x \in N : x^2 < 25\}$
   (iii) $\{x \in N : x$ is a prime number, $10 < x < 20\}$
   (iv) $\{x \in N : x = 2n, n \in N\}$
   (v) $\{x \in R : x > x\}$
   (vi) $\{x : x$ is a prime number which is a divisor of $60\}$
   (vii) $\{x : x$ is a two digit number such that the sum of its digits is $8\}$
   (viii) The set of all letters in the word 'Trigonometry'.
   (ix) The set of all letters in the word 'Better'.
2. Describe the following sets in set-builder form:
   (i) $A = \{1, 2, 3, 4, 5, 6\}$
   (ii) $B = \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$
   (iii) $C = \{0, 3, 6, 9, 12, \ldots\}$
   (iv) $D = \{10, 11, 12, 13, 14, 15\}$
   (v) $E = \{0\}$
   (vi) $\{1, 4, 9, 16, \ldots, 100\}$
   (vii) $\{2, 4, 6, 8, \ldots\}$
   (viii) $\{5, 25, 125, 625\}$
3. List all the elements of the following sets:
   (i) $A = \{x : x^2 \le 10, x \in Z\}$
   (ii) $B = \left\{x : x = \frac{1}{2n-1}, 1 \le n \le 5\right\}$
   (iii) $C = \left\{x : x \text{ is an integer}, -\frac{1}{2} < x < \frac{9}{2}\right\}$
   (iv) $D = \{x : x$ is a vowel in the word "EQUATION"$\}$
   (v) $E = \{x : x$ is a month of a year not having 31 days$\}$
   (vi) $F = \{x : x$ is a letter of the word "MISSISSIPPI"$\}$
4. Match each of the sets on the left in the roster form with the same set on the right described in the set-builder form:
   (i) $\{A, P, L, E\}$ (a) $\{x : x + 5 = 5, x \in Z\}$
   (ii) $\{5, -5\}$ (b) $\{x : x$ is a prime natural number and a divisor of $10\}$

(iii) $\{0\}$ (iii) $\{x : x$ is a letter of the word "RAJASTHAN"$\}$

(iv) $\{1, 2, 5, 10\}$ (iv) $\{x : x$ is a natural number and divisor of $10\}$

(v) $\{A, H, J, R, S, T, N\}$ (v) $\{x : x^2 - 25 = 0\}$

(vi) $\{2, 5\}$ (vi) $\{x : x$ is a letter of the word "APPLE"$\}$

5. Write the set of all vowels in the English alphabet which precede q.
6. Write the set of all positive integers whose cube is odd.
7. Write the set $\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}\right\}$ in the set-builder form.

ANSWERS

1. (i) $\{a, b, c, d\}$ (ii) $\{1, 2, 3, 4\}$ (iii) $\{11, 13, 17, 19\}$ (iv) $\{2, 4, 6, 8, \ldots\}$ (v) ϕ
   (vi) $\{2, 3, 5\}$ (vii) $\{17, 26, 35, 44, 53, 62, 71, 80\}$ (viii) $\{T, R, I, G, O, N, M, E, Y\}$
   (ix) $\{B, E, T, R\}$
2. (i) $\{x : x \in N, x < 7\}$ (ii) $\{x : x = 1/n, x \in N\}$ (iii) $\{x : x = 3n, n \in Z^+\}$
   (iv) $\{x : x \in N, 9 < x < 16\}$ (v) $\{x : x = 0\}$ (vi) $\{x^2 : x \in N, 1 \le x \le 10\}$
   (vii) $\{x : x = 2n, n \in N\}$ (viii) $\{5^n : n \in N, 1 \le n \le 4\}$
3. (i) $A = \{0, \pm 1, \pm 2, \pm 3\}$ (ii) $B = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\right\}$ (iii) $C = \{0, 1, 2, 3, 4\}$
   (iv) $D = \{A, E, I, O, U\}$ (v) $E = \{$Feb., April, June, Sept., November$\}$
   (vi) $F = \{M, I, S, P\}$
4. (i) $\to$ (vi) ; (ii) $\to$ (v); (iii) $\to$ (i) ; (iv) $\to$ (iv) ; (v) $\to$ (iii) ; (vi) $\to$ (ii)
5. $\{a, e, i, o\}$ 6. $\{2n + 1 : n \in Z, n \ge 0\}$ 7. $\left\{\frac{n}{n^2+1} : n \in N, n \le 7\right\}$

1.3 TYPES OF SETS

EMPTY SET *A set is said to be empty or null or void set if it has no element and it is denoted by ϕ.*

In Roster method, ϕ is denoted by $\{\ \}$.

It follows from this definition that a set A is an empty set if the statement $x \in A$ is not true for any x.

ILLUSTRATION 1 $\{x \in R : x^2 = -2\} = \phi$

ILLUSTRATION 2 $\{x \in N : 5 < x < 6\} = \phi$.

ILLUSTRATION 3 *The set A given by $A = \{x : x$ is an even prime number greater than $2\}$ is an empty set because 2 is the only even prime number.*

A set consisting of at least one element is called a non-empty or non-void set.

NOTE *If A and B are any two empty sets, then $x \in A$ iff (if and only if) $x \in B$ is satisfied because there is no element x in either A or B to which the condition may be applied. Thus, $A = B$. Hence, there is only one empty set and we denote it by ϕ. Therefore, article 'the' is used before empty set.*

SINGLETON SET *A set consisting of a single element is called a singleton set.*

ILLUSTRATION 4 *The set $\{5\}$ is a singleton set.*

ILLUSTRATION 5 *The set $\{x : x \in N$ and $x^2 = 9\}$ is a singleton set equal to $\{3\}$.*

FINITE SET *A set is called a finite set if it is either void set or its elements can be listed (counted, labelled) by natural numbers 1, 2, 3, ... and the process of listing terminates at a certain natural number n (say).*

CARDINAL NUMBER OF A FINITE SET *The number n in the above definition is called the cardinal number or order of a finite set A and is denoted by $n(A)$.*

INFINITE SET *A set whose elements cannot be listed by the natural numbers 1, 2, 3, ..., n, for any natural number n is called an infinite set.*

ILLUSTRATION 6 *Each one of the following sets is a finite set:*

(i) *Set of even natural numbers less than 100.* (ii) *Set of soldiers in Indian army.*

(iii) *Set of even prime natural numbers.* (iv) *Set of all persons on the earth.*

ILLUSTRATION 7 *Each one of the following sets is an infinite set:*

(i) *Set of all points in a plane.* (ii) *Set of all lines in a plane.*

(iii) $\{x \in R : 0 < x < 1\}$.

EQUIVALENT SETS *Two finite sets A and B are equivalent if their cardinal numbers are same i.e. $n(A) = n(B)$.*

EQUAL SETS *Two sets A and B are said to be equal if every element of A is a member of B, and every element of B is a member of A.*

If sets A and B are equal, we write $A = B$ and $A \ne B$ when A and B are not equal.

If $A = \{1, 2, 5, 6\}$ and $B = \{5, 6, 2, 1\}$. Then $A = B$, because each element of A is an element of B and vice-versa. Note that the elements of a set may be listed in any order.

It follows from the above definition and the definition of equivalent sets that equal sets are equivalent but equivalent sets need not be equal.

For example, $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ are equivalent sets but not equal sets.

ILLUSTRATIVE EXAMPLES

Type I **ON IDENTIFYING WHETHER GIVEN SET IS EMPTY OR NOT**

EXAMPLE 1 *Which of the following sets are empty sets?*

(i) $A = \{x : x^2 - 3 = 0$ *and x is rational*$\}$

(ii) $B = \{x : x$ *is an even prime number*$\}$

(iii) $C = \{x : 4 < x < 5, x \in N\}$

(iv) $D = \{x : x^2 = 25$, *and x is an odd integer*$\}$

SOLUTION (i) We know that there is no rational number whose square is 3. So, $x^2 - 3 = 0$ is not satisfied by any rational number. Hence, A is an empty set.

(ii) We know that 2 is the only even prime number. Therefore, $B = \{2\}$. So, B is not an empty set.

(iii) Since there is no natural number between 4 and 5. So, C is an empty set.

(iv) Since $x = 5, -5$ satisfy $x^2 = 25$ and ± 5 are odd integers. Therefore, $D = \{-5, 5\}$. Thus, D is a non-empty set.

Type II **ON EQUAL SETS**

EXAMPLE 2 *Find the pairs of equal sets, from the following sets, if any, giving reasons:*

$A = \{0\}$, $B = \{x : x > 15$ *and* $x < 5\}$, $C = \{x : x - 5 = 0\}$, $D = \{x : x^2 = 25\}$ [NCERT]

$E = \{x : x$ *is an integral positive root of the equation* $x^2 - 2x - 15 = 0\}$.

SOLUTION We have,

$A = \{0\}$,
$B = \{x : x > 15 \text{ and } x < 5\} = \phi$,
$C = \{x : x - 5 = 0\} = \{5\}$,
$D = \{x : x^2 = 25\} = \{-5, 5\}$,
and, $E = \{5\}$.
Clearly, $C = E$.

EXAMPLE 3 *Which of the following pairs of sets are equal? Justify your answer.*
(i) $A = \{x : x$ *is a letter in the word "LOYAL"*$\}$,
$B = \{x : x$ *is a letter of the word "ALLOY"*$\}$
(ii) $A = \{x : x \in Z \text{ and } x^2 \le 8\}$, $B = \{x : x \in R \text{ and } x^2 - 4x + 3 = 0\}$

SOLUTION We have,
(i) $A = \{L, O, Y, A, L\} = \{L, O, Y, A\}$
and, $B = \{A, L, L, O, Y\} = \{L, O, Y, A\}$
Clearly, $A = B$.
(ii) $A = \{x : x \in Z \text{ and } x^2 \le 8\} = \{-2, -1, 0, 1, 2\}$
and, $B = \{x : x \in R \text{ and } x^2 - 4x + 3 = 0\} = \{1, 3\}$
Since $0 \in A$ but $0 \notin B$.
So, $A \ne B$.

Type III ON FINITE AND INFINITE SETS

EXAMPLE 4 *State which of the following sets are finite and which are infinite:*
(i) $A = \{x : x \in Z \text{ and } x^2 - 5x + 6 = 0\}$
(ii) $B = \{x : x \in Z \text{ and } x^2 \text{ is even}\}$
(iii) $C = \{x : x \in Z \text{ and } x^2 = 36\}$
(iv) $D = \{x : x \in Z \text{ and } x > -10\}$

SOLUTION We have,
(i) $A = \{x : x \in Z \text{ and } x^2 - 5x + 6 = 0\} = \{2, 3\}$
So, A is a finite set
(ii) $B = \{x : x \in Z \text{ and } x^2 \text{ is even}\}$
$= \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$
Clearly, B is an infinite set.
(iii) $C = \{x : x \in Z \text{ and } x^2 = 36\}$
$= \{6, -6\}$
Clearly, A is a finite set.
(iv) $D = \{x : x \in Z \text{ and } x > -10\}$
$= \{-9, -8, -7, ...\}$
Clearly, D is an infinite set.

EXERCISE 1.3

1. Which of the following are examples of empty set?
   (i) Set of all even natural numbers divisible by 5;
   (ii) Set of all even prime numbers;
   (iii) $\{x : x^2 - 2 = 0$ and x is rational$\}$;

   (iv) $\{x : x$ is a natural number, $x < 8$ and simultaneously $x > 12\}$;
   (v) $\{x : x$ is a point common to any two parallel lines$\}$.
2. Which of the following sets are finite and which are infinite?
   (i) Set of concentric circles in a plane;
   (ii) Set of letters of the English Alphabets;
   (iii) $\{x \in N : x > 5\}$;
   (iv) $\{x \in N : x < 200\}$;
   (v) $\{x \in Z : x < 5\}$;
   (vi) $\{x \in R : 0 < x < 1\}$.
3. Which of the following sets are equal?
   (i) $A = \{1, 2, 3\}$;
   (ii) $B = \{x \in R : x^2 - 2x + 1 = 0\}$;
   (iii) $C = \{1, 2, 2, 3\}$;
   (iv) $D = \{x \in R : x^3 - 6x^2 + 11x - 6 = 0\}$.
4. Are the following sets equal?
   $A = \{x : x$ is a letter in the word reap$\}$;
   $B = \{x : x$ is a letter in the word paper$\}$;
   $C = \{x : x$ is a letter in the word rope$\}$.
5. From the sets given below, pair the equivalent sets:
   $A = \{1, 2, 3\}$, $B = \{t, p, q, r, s\}$, $C = \{\alpha, \beta, \gamma\}$, $D = \{a, e, i, o, u\}$.
6. Are the following pairs of sets equal? Give reasons.
   (i) $A = \{2, 3\}$, $B = \{x : x$ is a solution of $x^2 + 5x + 6 = 0\}$;
   (ii) $A = \{x : x$ is a letter of the word "WOLF"$\}$; [NCERT]
   $B = \{x : x$ is a letter of the word "FOLLOW"$\}$.
7. From the sets given below, select equal sets and equivalent sets.
   $A = \{0, a\}$, $B = \{1, 2, 3, 4\}$ $C = \{4, 8, 12\}$, $D = \{3, 1, 2, 4\}$,
   $E = \{1, 0\}$, $F = \{8, 4, 12\}$ $G = \{1, 5, 7, 11\}$, $H = \{a, b\}$.
8. Which of the following sets are equal?
   $A = \{x : x \in N, x < 3\}$,
   $B = \{1, 2\}$
   $C = \{3, 1\}$
   $D = \{x : x \in N, x$ is odd, $x < 5\}$,
   $E = \{1, 2, 1, 1\}$ $F = \{1, 1, 3\}$.
9. Show that the set of letters needed to spell "CATARACT" and the set of letters needed to spell "TRACT" are equal. [NCERT]

ANSWERS

1. (iii), (iv), (v) 2. (i) Infinite (ii) finite (iii) Infinite (iv) Finite (v) Infinite (vi) Infinite.
3. $A = C = D$ 4. No 5. $A, C; B, D$ 6. (i) No (ii) Yes
7. Equal sets : $B = D, C = F$ Equivalent sets : $A, E, H; B, D, G; C, F$
8. $A = B = E$, $C = D = F$

HINTS TO NCERT & SELECTED PROBLEMS

6. (ii) We have,
   $A = \{x : x$ is a letter of the word "WOLF"$\} = \{W, O, L, F\}$

B = {x : x is a letter of the word "FOLLOW"} = {W, O, L, F}

Clearly, $A = B$.

9. A = Set of letters of the word "CATARACT" = {A, C, R, T}

B = Set of letters of the word "TRACT" = {A, C, R, T}

Clearly, $A = B$.

1.4 SUBSETS

SUBSETS *Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B.*

If A is a subset of B, we write $A \subseteq B$, which is read as "A is a subset of B" or "A is contained in B".

Thus, $A \subseteq B$ if $a \in A \Rightarrow a \in B$.

The symbol "$\Rightarrow$" stands for "implies".

If A is a subset of B, we say that B contains A or B is a super set of A and we write $B \supseteq A$.

If A is not a subset of B, we write $A \not\subseteq B$.

Obviously, every set is a subset of itself and the empty set is subset of every set. These two subsets are called *improper subsets*. A subset A of a set B is called a *proper subset* of B if $A \neq B$ and we write $A \subset B$. In such a case, we also say that B is a super set of A. Thus, if A is a proper subset of B, then there exists an element $x \in B$ such that $x \notin A$.

It follows immediately from this definition and the definition of equal sets that two sets A and B are equal iff $A \subseteq B$ and $B \subseteq A$.

Thus, whenever we want to prove that two sets A and B are equal, we must prove that $A \subseteq B$ and $B \subseteq A$.

ILLUSTRATION 1 *Clearly* $\{1\} \subseteq \{1, 2, 3\}$, *but* $\{1, 4\} \not\subseteq \{1, 2, 3\}$.

ILLUSTRATION 2 *Clearly,* $N \subset Z \subset Q \subset R \subset C$.

ILLUSTRATION 3 *If A is the set of all divisors of 68 and B is the set of all prime divisors of 68, then B is the subset of A and we write* $B \subset A$.

1.4.1 SOME RESULTS ON SUBSETS

THEOREM 1 *Every set is a subset of itself.*

PROOF Let A be any set. Then, each element of A is clearly in A itself. Hence, $A \subseteq A$.

THEOREM 2 *The empty set is a subset of every set.*

PROOF Let A be any set and ϕ be the empty set. In order to show that $\phi \subseteq A$, we must show that every element of ϕ is an element of A also. But, ϕ contains no element. So, every element of ϕ is in A.

Hence, $\phi \subseteq A$.

THEOREM 3 *The total number of subsets of a finite set containing n elements is* 2^n.

PROOF Let A be a finite set containing n elements. Let $0 \le r \le n$. Consider those subsets of A that have r elements each. We know that the number of ways in which r elements can be chosen out of n elements is ${}^nC_r$. Therefore, the number of subsets of A having r elements each is ${}^nC_r$.

Hence, the total number of subsets of A

$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = (1+1)^n = 2^n \quad \text{[Using binomial theorem]}$$

ILLUSTRATION 1 *Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of m and n.*

SOLUTION Let A and B be two sets having m and n elements respectively. Then,

Number of subsets of set $A = 2^m$, Number of subsets of set $B = 2^n$.

It is given that, $2^m - 2^n = 56$

$$\Rightarrow \quad 2^n(2^{m-n} - 1) = 2^3(2^3 - 1)$$

$$\Rightarrow \quad n = 3 \text{ and } m - n = 3 \Rightarrow n = 3 \text{ and } m = 6.$$

ILLUSTRATION 2 *If* $X = \{4^n - 3n - 1 : n \in N\}$ *and* $Y = \{9(n-1) : n \in N\}$, *prove that* $X \subseteq Y$.

SOLUTION Let $x_n = 4^n - 3n - 1$, $n \in N$. Then, $x_1 = 4 - 3 - 1 = 0$.

And, for any $n \ge 2$, we have

$$x_n = 4^n - 3n - 1 = (1+3)^n - 3n - 1$$

$$\Rightarrow \quad x_n = {}^nC_0 + {}^nC_1 . 3 + {}^nC_2 3^2 + {}^nC_3 . 3^3 + \dots + {}^nC_n . 3^n - 3n - 1$$

[Using Binomial Theorem]

$$\Rightarrow \quad x_n = 1 + 3n + {}^nC_2 . 3^2 + {}^nC_3 . 3^3 + \dots + {}^nC_n . 3^n - 3n - 1 \quad [\because {}^nC_0 = 1, {}^nC_1 = n]$$

$$\Rightarrow \quad x_n = 3^2\{{}^nC_2 + {}^nC_3 . 3 + {}^nC_4 . 3^2 + \dots + {}^nC_n . 3^{n-2}\}$$

$$\Rightarrow \quad x_n = 9\{{}^nC_2 + {}^nC_3 . 3 + {}^nC_4 . 3^2 + \dots + {}^nC_n . 3^{n-2}\}$$

Thus, x_n is some positive integral multiple of 9 for all $n \ge 2$.

Thus, X consists of all those positive integral multiples of 9 which are of the form $9\{{}^nC_2 + 3 . {}^nC_3 + 3^2 . {}^nC_4 + \dots + 3^{n-2} . {}^nC_n\}$ together with 0.

And, $Y = \{9(n-1) : n \in N\}$ implies that it consists of all integral multiples of 9 together with 0.

Hence, $X \subseteq Y$.

1.4.2 SUBSETS OF THE SET R OF REAL NUMBERS

Following sets are important subsets of the set R of all real numbers:

(i) The set of all natural numbers $N = \{1, 2, 3, 4, 5, 6, \dots\}$

(ii) The set of all integers $Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

(iii) The set of all rational numbers $Q = \{x : x = \frac{m}{n}, m, n \in Z, n \neq 0\}$.

(iv) The set of all irrational numbers. It is denoted by T.

Thus,

$$T = \{x : x \in R \text{ and } x \notin Q\}.$$

Clearly, $N \subset Z \subset Q \subset R$, $T \subset R$ and $N \not\subset T$.

1.4.3 INTERVALS AS SUBSETS OF R

On real line various types of infinite subsets are designated as intervals as defined below:

CLOSED INTERVAL *Let a and b be two given real numbers such that* $a < b$. *Then, the set of all real numbers x such that* $a \le x \le b$ *is called a closed interval and is denoted by* $[a, b]$.

Thus, $[a, b] = \{x \in R : a \le x \le b\}$.

On the real line, $[a, b]$ may be graphed as shown in Fig. 1.1

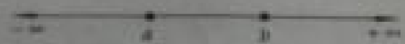


Fig. 1.1

For example, $[-1, 2] = \{x \in R : -1 \leq x \leq 2\}$ is the set of all real numbers lying between -1 and 2 including the end points. Clearly, it is an infinite subset of R.

OPEN INTERVAL *If a and b are two real numbers such that $a < b$, then the set of all real numbers x satisfying $a < x < b$ is called an open interval and is denoted by (a, b) or $]a, b[$.*

Thus, $(a, b) = \{x \in R : a < x < b\}$

On the real line, (a, b) may be graphed as shown in Fig. 1.2.

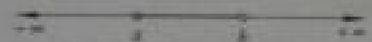


Fig. 1.2

Encircling a and b means that a and b are not included in the set.

For example, $(1, 2) = \{x \in R : 1 < x < 2\}$ is the set of all real numbers lying between 1 and 2 excluding the end-points 1 and 2. This is an infinite subset of R.

SEMI-OPEN OR SEMI-CLOSED INTERVAL *If a and b are two real numbers such that $a < b$, then the sets $(a, b] = \{x \in R : a < x \leq b\}$ and $[a, b) = \{x \in R : a \leq x < b\}$ are known as semi-open or semi-closed intervals. $(a, b]$ and $[a, b)$ are also denoted by $]a, b]$ and $[a, b[$ respectively.*

On real line these sets may be graphed as shown in Figs. 1.3 and 1.4 respectively.

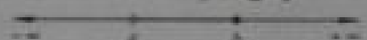


Fig. 1.3

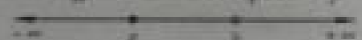


Fig. 1.4

The number $b - a$ is called the length of any of the intervals (a, b), $[a, b]$, $[a, b)$ and $(a, b]$.

These notations provide an alternative way of designating the subsets of the set R of all real numbers. For example, the interval $[0, \infty)$ denotes the set R^+ of all non-negative real numbers, while the interval $(-\infty, 0)$ denotes the set R^- of all negative real numbers. The interval $(-\infty, \infty)$ denotes the set R of all real numbers.

1.5 UNIVERSAL SET

In any discussion in set theory, there always happens to be a set that contains all sets under consideration i.e. it is a super set of each of the given sets. Such a set is called the universal set and is denoted by U.

Thus, a set that contains all sets in a given context is called the universal set.

ILLUSTRATION 1 *When we study two dimensional coordinate geometry, then the set of all points in xy-plane is the universal set.*

ILLUSTRATION 2 *When we are using sets containing natural numbers, then N is the universal set.*

ILLUSTRATION 3 *If $A = \{1, 2, 3\}$, $B = \{2, 4, 5, 6\}$ and $C = \{1, 3, 5, 7\}$, then $U = \{1, 2, 3, 4, 5, 6, 7\}$ can be taken as the universal set.*

ILLUSTRATION 4 *When we are using intervals on real line, the set R of real numbers is taken as the universal set.*

1.6 POWER SET

POWER SET *Let A be a set. Then the collection or family of all subsets of A is called the power set of A and is denoted by $P(A)$.*

That is, $P(A) = \{S : S \subset A\}$.

Since the empty set and the set A itself are subsets of A and are therefore elements of $P(A)$. Thus, the power set of a given set is always non-empty.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$. Then, the subsets of A are :*

$\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ and $\{1, 2, 3\}$.

Hence, $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

ILLUSTRATION 2 *If A is the void set ϕ, then $P(A)$ has just one element ϕ i.e. $P(\phi) = \{\phi\}$.*

ILLUSTRATION 3 *Show that $n(P[P(P(\phi))]) = 4$.*

SOLUTION We have,

$P(\phi) = \{\phi\}$

$\therefore$ $P(P(\phi)) = \{\phi, \{\phi\}\}$

$\Rightarrow$ $P[P(P(\phi))] = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Hence, $P[P(P(\phi))]$ consists of 4 elements i.e. $n\{P[P(P(\phi))]\} = 4$.

We know that a set having n elements has 2^n subsets. Therefore, if A is a finite set having n elements, then $P(A)$ has 2^n elements.

ILLUSTRATION 4 *If $A = \{a, \{b\}\}$, find $P(A)$.*

SOLUTION Let $B = \{b\}$. Then, $A = \{a, B\}$.

$\therefore$ $P(A) = \{\phi, \{a\}, \{B\}, \{a, B\}\} = \{\phi, \{a\}, \{\{b\}\}, \{a, \{b\}\}\}$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Consider the following sets:*

ϕ, $A = \{1, 2\}$, $B = \{1, 4, 8\}$, $C = \{1, 2, 4, 6, 8\}$

Insert the correct symbol $\subset$ or $\not\subset$ between each of the following pair of sets:

(i) $\phi \ldots B$ (ii) $A \ldots B$ (iii) $A \ldots C$ (iv) $B \ldots C$

SOLUTION

(i) Since null set is subset of every set.

$\therefore$ $\phi \subset B$

(ii) Clearly, $2 \in A$ but $2 \notin B$. So, $A \not\subset B$.

(iii) Since all elements of set A are in C and $A \neq C$. So, $A \subset C$.

(iv) Clearly, all elements of set B are in set C and $B \neq C$. So, $B \subset C$.

EXAMPLE 2 *Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$ and $C = \{b, d\}$. Find all sets X such that :*

(i) $X \subset B$ and $X \subset C$ (ii) $X \subset A$ and $X \not\subset B$.

SOLUTION We have,

(i) $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \ldots\ldots\}$

$P(B) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

and, $P(C) = \{\phi, \{b\}, \{d\}, \{b, d\}\}$

Now, $X \subset B$ and $X \subset C$

$\Rightarrow$ $X \in P(B)$ and $X \in P(C)$.

$\Rightarrow$ $X = \phi, \{b\}$

(iii) We have,

$X \subset A$ and $X \not\subset B$

$\Rightarrow$ X is a subset of A but X is not a subset of B

$\Rightarrow$ $X \in P(A)$ but $X \notin P(B)$

$\Rightarrow$ $X = \{d\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}$.

EXAMPLE 3 *Let A, B and C be three sets. If $A \in B$ and $B \subset C$, is it true that $A \subset C$? If not give an example.*

SOLUTION Consider the following sets:

$A = \{a\}$, $B = \{\{a\}, b\}$ and $C = \{\{a\}, b, c\}$.

Clearly, $A \in B$ and $B \subset C$. But, $A \not\subset C$ as $a \in A$ but $a \notin C$.

Thus, the given statement is not true.

EXAMPLE 4 *Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ and $C = \{2, 4\}$. Find all sets X satisfying each pair of conditions:*

(i) $X \subset B$ and $X \not\subset C$ (ii) $X \subset B$, $X \neq B$ and $X \not\subset C$ (iii) $X \subset A$, $X \subset B$ and $X \subset C$.

SOLUTION We have,

(i) $X \subset B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B but X is not a subset of C

$\Rightarrow$ $X \in P(B)$ but $x \notin P(C)$

$\Rightarrow$ $X = \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

(ii) We have,

$X \subset B$, $X \neq B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B other than B itself and X is not a subset of C

$\Rightarrow$ $X \in P(B)$, $X \notin P(C)$ and $X \neq B$

$\Rightarrow$ $X = \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$

(iii) We have,

$X \subset A$, $X \subset B$ and $X \subset C$

$\Rightarrow$ $X \in P(A)$, $X \in P(B)$ and $X \in P(C)$

$\Rightarrow$ X is a subset of A, B and C

$\Rightarrow$ $X = \phi, \{2\}$

EXAMPLE 5 *Let B be a subset of a set A and let $P(A : B) = \{X \in P(A) : X \supset B\}$.*

(i) Show that: $P(A : \phi) = P(A)$

(ii) If $A = \{a, b, c, d\}$ and $B = \{a, b\}$. List all the members of the set $P(A : B)$.

SOLUTION (i) We have,

$P(A : B) = \{X \in P(A) : X \supset B\}$

$= \{X \in P(A) : B \subset X\}$

$=$ Set of all those subsets of A which contain B

$\therefore$ $P(A : \phi) =$ Set of all those subsets of A which contain ϕ

$=$ Set of all subsets of set A

$= P(A)$.

(ii) If $A = \{a, b, c, d\}$ and $B = \{a, b\}$. Then,

$P(A : B) =$ Set of all those subsets of set A which contain B

$= \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}$

EXAMPLE 6 *Prove that $A \subset \phi$ implies $A = \phi$.*

SOLUTION We know that two sets A and B are equal iff $A \subset B$ and $B \subset A$. Also, We know that

$\phi \subset A$

and, $A \subset \phi$ [Given]

$\therefore$ $A = \phi$

EXAMPLE 7 *In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.*

(i) If $x \in A$ and $A \in B$, then $x \in B$ *(ii) If $A \subset B$ and $B \in C$, then $A \in C$*

(iii) If $A \subset B$ and $B \subset C$, then $A \subset C$ *(iv) If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$*

(v) If $x \in A$ and $A \not\subset B$, then $x \in B$ *(vi) If $A \subset B$ and $x \notin B$, then $x \notin A$*

SOLUTION (i) False.

Consider $A = \{1\}$, $B = \{\{1\}, 2\}$.

Clearly $1 \in A$ and $A \in B$ but $1 \notin B$.

So, $x \in A$ and $A \in B$ need not imply that $x \in B$.

(ii) False.

Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{\{1, 2\}, 3\}$. Then, we observe that

$A \subset B$ and $B \in C$ but $A \notin C$.

Thus, $A \subset B$ and $B \in C$ need not imply that $A \in C$.

(iii) True.

Let $x \in A$. Then,

$A \subset B \Rightarrow x \in B$

$\Rightarrow x \in C$ $[\because B \subset C]$

Thus, $x \in A \Rightarrow x \in C$ for all $x \in A \Rightarrow A \subset C$.

Hence, $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

(iv) False.

Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 2, 5\}$

Then, $A \not\subset B$ and $B \not\subset C$. But $A \subset C$.

Thus, $A \not\subset B$ and $B \not\subset C$ need not imply that $A \not\subset C$

(v) False.

Let $A = \{1, 2\}$ and $B = \{2, 3, 4, 5\}$. Then, we observe that

$1 \in A$ and $A \not\subset B$ but $1 \notin B$.

Thus, $x \in A$ and $A \not\subset B$ need not imply that $x \in B$.

(vi) True.

Let $A \subset B$. Then, we observe that

$x \in A \Rightarrow x \in B \Leftrightarrow x \notin B \Rightarrow x \notin A$.

EXAMPLE 8 *Write the following subsets of R as intervals:*

(i) $\{x : x \in R, -4 < x \leq 6\}$ (ii) $\{x : x \in R, -12 < x < -10\}$

(iii) $\{x : x \in R, 0 \leq x < 7\}$ (iv) $\{x : x \in R, 3 \leq x \leq 4\}$.

Also, find the length of each interval.

SOLUTION We have,

(i) $\{x : x \in R, -4 < x \leq 6\} = (-4, 6]$. Length $= 6 - (-4) = 10$

(ii) $\{x : x \in R, -12 < x < -10\} = (-12, -10)$. Length $= -10 - (-12) = 2$

(iii) $\{x : x \in R, 0 \leq x < 7\} = [0, 7)$. Length $= 7 - 0 = 7$

(iv) $\{x : x \in R, 3 \leq x \leq 4\} = [3, 4]$. Length $= 4 - 3 = 1$

EXAMPLE 9 *Write the following intervals in the set-builder form:*

(i) $(-7, 0)$ (ii) $[6, 12]$ (iii) $(6, 12]$ (iv) $[-20, 3)$

SOLUTION We have,

(i) $(-7, 0) = \{x : x \in R \text{ and } -7 < x < 0\}$

(ii) $[6, 12] = \{x : x \in R \text{ and } 6 \leq x \leq 12\}$

(iii) $(6, 12] = \{x : x \in R \text{ and } 6 < x \leq 12\}$

(iv) $[-20, 3) = \{x : x \in R \text{ and } -20 \leq x < 3\}$

EXAMPLE 10 *Let $A = \{1, 2, \{3, 4\}, 5\}$. Which of the following statements are incorrect and why?*

(i) $\{3, 4\} \subset A$ (ii) $\{3, 4\} \in A$ (iii) $\{\{3, 4\}\} \subset A$ (iv) $1 \in A$

(v) $1 \subset A$ (vi) $\{1, 2, 5\} \subset A$ (vii) $\{1, 2, 5\} \in A$ (viii) $\{1, 2, 3\} \subset A$

(ix) $\phi \in A$ (x) $\phi \subset A$ (xi) $\{\phi\} \subset A$

SOLUTION $\{3, 4\}$ is an element of set A. Therefore, $\{3, 4\} \in A$ is correct and $\{3, 4\} \subset A$ is incorrect.

So, (i) is incorrect and (ii) is correct.

As $\{3, 4\}$ is an element of set A. Therefore, $\{\{3, 4\}\}$ is a set containing element $\{3, 4\}$ which belongs to A.

So, $\{\{3, 4\}\} \subset A$.

Hence, (iii) is correct.

Since 1 is an element of set A. So, $1 \in A$ is correct and $1 \subset A$ is incorrect.

So, (iv) is correct and (v) is incorrect.

Since 1, 2, 5 are elements of set A. Therefore, $\{1, 2, 5\}$ is a subset of set A.

Hence, (vi) is correct and (vii) is incorrect.

As 3 is not an element of set A. So, $\{1, 2, 3\} \subset A$ is incorrect.

The null set is subset of every set.

So, $\phi \subset A$ is correct and $\phi \in A$ is incorrect.

Hence, (ix) is incorrect and (x) is correct.

As $\phi \subset A$ but $\{\phi\}$ is not a subset of A. So, (xi) is incorrect.

EXERCISE 1.4

1. Which of the following statements are true ? Give reason to support your answer.
   (i) For any two sets A and B either $A \subseteq B$ or $B \subseteq A$;
   (ii) Every subset of an infinite set is infinite;
   (iii) Every subset of a finite set is finite;
   (iv) Every set has a proper subset;
   (v) $\{a, b, a, b, a, b, \ldots\}$ is an infinite set;
   (vi) $\{a, b, c\}$ and $\{1, 2, 3\}$ are equivalent sets;
   (vii) A set can have infinitely many subsets.
2. State whether the following statements are true or false:
   (i) $1 \in \{1, 2, 3\}$ (ii) $a \subset \{b, c, a\}$
   (iii) $\{a\} \in \{a, b, c\}$ (iv) $\{a, b\} = \{a, a, b, b, a\}$
   (v) The set $\{x : x + 8 = 8\}$ is the null set.
3. Decide among the following sets, which are subsets of which:
   $A = \{x : x \text{ satisfies } x^2 - 8x + 12 = 0\}$,
   $B = \{2, 4, 6\}$, $C = \{2, 4, 6, 8, \ldots\}$, $D = \{6\}$.

4. Write which of the following statements are true ? Justify your answer.
   (i) The set of all integers is contained in the set of all rational numbers.
   (ii) The set of all crows is contained in the set of all birds.
   (iii) The set of all rectangles is contained in the set of all squares.
   (iv) The set of all real numbers is contained in the set of all complex numbers.
   (v) The sets $P = \{a\}$ and $B = \{\{a\}\}$ are equal.
   (vi) The sets $A = \{x : x$ is a letter of the word "LITTLE"$\}$ and,
   $B = \{x : x$ is a letter of the word "TITLE"$\}$ are equal.
5. Which of the following statements are correct?
   Write a correct form of each of the incorrect statements.
   (i) $a \subset \{a, b, c\}$ (ii) $\{a\} \in \{a, b, c\}$ (iii) $a \in \{\{a\}, b\}$
   (iv) $\{a\} \subset \{\{a\}, b\}$ (v) $\{b, c\} \subset \{a, \{b, c\}\}$ (vi) $\{a, b\} \subset \{a, \{b, c\}\}$
   (vii) $\phi \in \{a, b\}$ (viii) $\phi \subset \{a, b, c\}$ (ix) $\{x : x + 3 = 3\} = \phi$
6. Let $A = \{a, b, \{c, d\}, e\}$. Which of the following statements are false and why?
   (i) $\{c, d\} \subset A$ (ii) $\{c, d\} \in A$ (iii) $\{\{c, d\}\} \subset A$
   (iv) $a \in A$ (v) $a \subset A$ (vi) $\{a, b, e\} \subset A$
   (vii) $\{a, b, e\} \in A$ (viii) $\{a, b, c\} \subset A$ (ix) $\phi \in A$
   (x) $\{\phi\} \subset A$
7. Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$. Determine which of the following is true or false:
   (i) $1 \in A$ (ii) $\{1, 2, 3\} \subset A$ (iii) $\{6, 7, 8\} \in A$
   (iv) $\{\{4, 5\}\} \subset A$ (v) $\phi \in A$ (vi) $\phi \subset A$
8. Let $A = \{\phi, \{\phi\}, 1, \{1, \phi\}, 2\}$. Which of the following are true?
   (i) $\phi \in A$ (ii) $\{\phi\} \in A$ (iii) $\{1\} \in A$
   (iv) $\{2, \phi\} \subset A$ (v) $2 \subset A$ (vi) $\{2, \{1\}\} \not\subset A$
   (vii) $\{\{2\}, \{1\}\} \not\subset A$ (viii) $\{\phi, \{\phi\}, \{1, \phi\}\} \subset A$ (ix) $\{\{\phi\}\} \subset A$
9. Write down all possible subsets of each of the following sets:
   (i) $\{a\}$, (ii) $\{0, 1\}$, (iii) $\{a, b, c\}$,
   (iv) $\{1, \{1\}\}$, (v) $\{\phi\}$.
10. Write down all possible proper subsets each of the following sets:
    (i) $\{1, 2\}$, (ii) $\{1, 2, 3\}$, (iii) $\{1\}$.
11. What is the total number of proper subsets of a set consisting of n elements?
12. If A is any set, prove that: $A \subseteq \phi \Leftrightarrow A = \phi$.
13. Prove that : $A \subseteq B$, $B \subseteq C$ and $C \subseteq A \Rightarrow A = C$.
14. How many elements has $P(A)$, if $A = \phi$?
15. What universal set (s) would you propose for each of the following:
    (i) The set of right triangles.
    (ii) The set of isosceles triangles.
16. If $X = \{8^n - 7n - 1 : n \in N\}$ and $Y = \{49(n-1) : n \in N\}$, then prove that $X \subseteq Y$.

ANSWERS

1. (i) F, $A = \{1, 2, 3\}$, $B = \{a, b\}$ (ii) F, $A = \{1, 2\}$ is a finite subset of N.
   (iii) T (iv) F, ϕ does not have a proper subset (v) F, Given set $= \{a, b\}$
   (vi) T (vii) F
2. (i) T (ii) F (iii) F (iv) T (v) F

3. $D \subset A \subset B \subset C$
4. (i) T (ii) T (iii) F (iv) T (v) F (vi) T
5. (i) $a \in \{a, b, c\}$ (ii) $\{a\} \subset \{a, b, c\}$ (iii) $\{a\} \in \{\{a\}, b\}$
(iv) $\{\{a\}\} \subset \{\{a\}, b\}$ (v) $\{b, c\} \in \{a, \{b, c\}\}$ (vi) $\{a, b\} \not\subset \{a, \{b, c\}\}$
(vii) $\phi \subset \{a, b\}$ (viii) $\phi \subset \{a, b, c\}$ (ix) $\{x : x + 3 = 3\} \neq \phi$
6. (i) F (ii) T (iii) T (iv) T (v) F (vi) T
(vii) F (viii) F (ix) F (x) F
7. (i) F (ii) F (iii) T (iv) T (v) F (vi) T
8. (i) T (ii) T (iii) F (iv) T (v) F (vi) T
(vii) T (viii) T (ix) T
9. (i) $\phi, \{a\}$ (ii) $\phi, \{0\}, \{1\}, \{0, 1\}$ (iii) $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$
(iv) $\phi, \{1\}, \{\{1\}\}, \{1, \{1\}\}$ (v) $\phi, \{\phi\}$.
10. (i) $\{1\}, \{2\}$ (ii) $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}$ (iii) No proper subset
11. $2^7 - 1$ 14. 1
15. (i) The set of all triangles in a plane. (ii) The set of all triangles in a plane.

HINTS TO SELECTED PROBLEM

16. Let $x_n = 8^n - 7n - 1 = (1 + 7)^n - 7n - 1 = {}^nC_2 7^2 + {}^nC_3 7^3 + \dots + {}^nC_n 7^n$
$= 49\,({}^nC_2 + {}^nC_3 . 7 + \dots + {}^nC_n 7^{n-2})$ for $n \geq 2$.
For $x = 1$, $x_n = 0$.
Thus, X contains all positive integral multiples of the form $49\,k_n$ of 49, where
$k_n = {}^nC_2 + {}^nC_3 . 7 + {}^nC_4 . 7^2 + \dots + {}^nC_n 7^{n-2}$
But, Y contains all positive integral multiples of 49 including zero. Thus, $X \subseteq Y$.

1.7 VENN DIAGRAMS

Sometimes pictures are very helpful in our thinking. First of all a Swiss mathematician Euler gave an idea to represent a set by the points in a closed curve. Later on British mathematician John-Venn (1834-1883) brought this idea to practice. That is why the diagrams drawn to represent sets are called *Venn-Euler diagrams* or simply Venn-diagrams. In Venn-diagrams the universal set U is represented by points within a rectangle and its subsets are represented by points in closed curves (usually circles) within the rectangle. If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B as shown in Fig. 1.5 (i). If A and B are not equal but they have some common elements, then to represent A and B we draw two intersecting circles. (See Fig. 1.5 (ii)). Two disjoint sets are represented by two non-intersecting circles. (See Fig. 1.5 (iii))

1.8 OPERATIONS ON SETS

In this section, we shall introduce some operations on sets to construct new sets from given ones.

Fig. 1.5 (i)

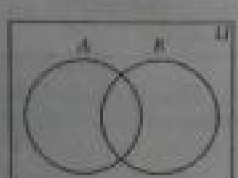


Fig. 1.5 (ii)

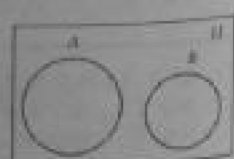


Fig. 1.5 (iii)

UNION OF SETS *Let A and B be two sets. The union of A and B is the set of all those elements which belong either to A or to B or to both A and B.*

We shall use the notation $A \cup B$ (read as "A union B") to denote the union of A and B.

Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Clearly, $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$.

And, $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$.

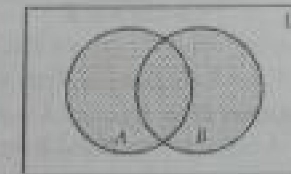


Fig. 1.6

In Fig. 1.6 the shaded part represents $A \cup B$. It is evident from the definition that $A \subseteq A \cup B$, $B \subseteq A \cup B$.

If A and B are two sets such that $A \subset B$, then $A \cup B = B$. Also, $A \cup B = A$, if $B \subset A$.

ILLUSTRATION 1 *If $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$, then $A \cup B = \{1, 2, 3, 5, 7\}$.*

ILLUSTRATION 2 *If $A = \{x : x = 2n + 1, n \in Z\}$ and $B = \{x : x = 2n, n \in Z\}$, then*

$A \cup B = \{x : x \text{ is an odd integer}\} \cup \{x : x \text{ is an even integer}\}$

$\Rightarrow$ $A \cup B = \{x : x \text{ is an integer}\} = Z$.

NOTE *If $A_1, A_2, \dots, A_n$ is a finite family of sets, then their union is denoted by $\bigcup_{i=1}^{n} A_i$ or, $A_1 \cup A_2 \cup A_3 \dots \cup A_n$.*

ILLUSTRATION 3 Let $A = \{1, 2, 3\}$, $B = \{3, 5\}$, $C = \{4, 7, 8\}$. Then,

$A \cup B \cup C = \{1, 2, 3, 4, 5, 7, 8\}$

INTERSECTION OF SETS *Let A and B be two sets. The intersection of A and B is the set of all those elements that belong to both A and B.*

The intersection of A and B is denoted by $A \cap B$ (read as "A intersection B")

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Clearly, $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

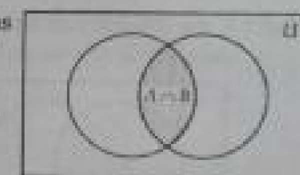


Fig. 1.7

In Fig. 1.7 the shaded region represents $A \cap B$.

Evidently, $A \cap B \subseteq A$, $A \cap B \subseteq B$.

NOTE *If $A_1, A_2, \dots, A_n$ is a finite family of sets, then their intersection is denoted by $\bigcap_{i=1}^{n} A_i$ or, $A_1 \cap A_2 \cap \dots \cap A_n$.*

ILLUSTRATION 4 *If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 9, 12\}$, then $A \cap B = \{1, 3\}$.*

ILLUSTRATION 5 *If $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{4, 6, 7, 8, 9, 10, 11\}$, then*

$A \cap B = \{2, 4, 6\}$

$\therefore$ $A \cap B \cap C = \{4, 6\}$

ILLUSTRATION 6 *If $A = \{x : x = 2n, n \in Z\}$ and $B = \{x : x = 3n, n \in Z\}$, then*

$A \cap B = \{x : x = 2n, n \in Z\} \cap \{x : x = 3n, n \in Z\}$

$\Rightarrow$ $A \cap B = \{\dots, -4, -2, 0, 2, 4, 6, \dots\} \cap \{\dots, -9, -6, -3, 0, 3, 6, 9, \dots\}$

$\Rightarrow$ $A \cap B = \{\dots, -6, 0, 6, 12, \dots\} = \{x : x = 6n, n \in Z\}$.

ILLUSTRATION 7 *If $A = \{x : x = 3n, n \in Z\}$ and $B = \{x : x = 4n, n \in Z\}$, then find $A \cap B$.*

SOLUTION We have,

$x \in A \cap B$

$\Leftrightarrow$ $x = 3n, n \in Z$ and $x = 4n, n \in Z$

$\Leftrightarrow$ x is a multiple of 3 and x is a multiple of 4

$\Leftrightarrow$ x is a multiple of 3 and 4 both

⇔ x is a multiple of 12.

⇔ $x = 12n, n \in Z$

Hence, $A \cap B = \{x : x = 12n, n \in Z\}$.

If A and B are two sets, then $A \cap B = A$, if $A \subset B$ and $A \cap B = B$, if $B \subset A$.

DISJOINT SETS *Two sets A and B are said to be disjoint, if* $A \cap B = \phi$.

If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.

Fig. 1.8

ILLUSTRATION 8 *If* $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{7, 8, 9, 10, 11\}$ *and* $C = \{6, 8, 10, 12, 14\}$, *then A and B are disjoint sets, while A and C are intersecting sets.*

DIFFERENCE OF SETS *Let A and B be two sets. The difference of A and B, written as* $A - B$, *is the set of all those elements of A which do not belong to B.*

Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

or, $A - B = \{x \in A : x \notin B\}$.

Clearly, $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$.

In Fig. 1.9, the shaded part represents $A - B$.

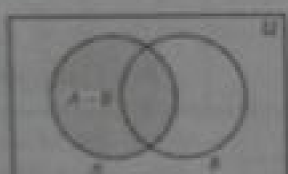


Fig. 1.9

Fig. 1.10

Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A.

i.e. $B - A = \{x \in B : x \notin A\}$.

In Fig. 1.10, the shaded part represents $B - A$.

ILLUSTRATION 9 *If* $A = \{2, 3, 4, 5, 6, 7\}$ *and* $B = \{3, 5, 7, 9, 11, 13\}$, *then* $A - B = \{2, 4, 6\}$ *and* $B - A = \{9, 11, 13\}$.

SYMMETRIC DIFFERENCE OF TWO SETS *Let A and B be two sets. The symmetric difference of sets A and B is the set* $(A - B) \cup (B - A)$ *and is denoted by* $A \Delta B$.

Thus, $A \Delta B = (A - B) \cup (B - A) = \{x : x \notin A \cap B\}$

The shaded part in Fig. 1.11 represents $A \Delta B$.

Fig. 1.11

ILLUSTRATION 10 *If* $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ *and* $B = \{1, 3, 5, 6, 7, 8, 9\}$, *then*

$A - B = \{2, 4\}$, $B - A = \{9\}$

∴ $A \Delta B = \{2, 4, 9\}$.

ILLUSTRATION 11 *If* $A = \{x \in R : 0 < x < 3\}$ *and* $B = \{x \in R : 1 \leq x \leq 5\}$, *then*

$A - B = \{x \in R : 0 < x < 1\}$, $B - A = \{x \in R : 3 \leq x \leq 5\}$

and $A \Delta B = \{x \in R : 0 < x < 1\} \cup \{x \in R : 3 \leq x \leq 5\} = \{x \in R : 0 < x < 1 \text{ or } 3 \leq x \leq 5\}$.

COMPLEMENT OF A SET *Let U be the universal set and let A be a set such that* $A \subset U$. *Then, the complement of A with respect to U is denoted by* A' *or* A^c *or* $U - A$ *and is defined the set of all those elements of U which are not in A.*

Thus $A' = \{x \in U : x \notin A\}$.

Clearly, $x \in A' \Leftrightarrow x \notin A$.

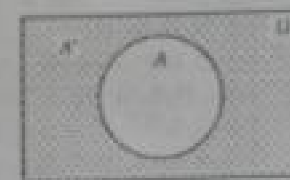


Fig. 1.12

ILLUSTRATION 12 *Let the set of natural numbers* $N = \{1, 2, 3, 4, ...\}$ *be the universal set and let* $A = \{2, 4, 6, 8, ...\}$. *Then* $A' = \{1, 3, 5, ...\}$.

ILLUSTRATION 13 *If* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *and* $A = \{1, 3, 5, 7, 9\}$, *then* $A' = \{2, 4, 6, 8\}$.

Following results are direct consequences of the definition of the complement of a set.

(i) $U' = \{x \in U : x \notin U\} = \phi$;

(ii) $\phi' = \{x \in U : x \notin \phi\} = U$;

(iii) $(A')' = \{x \in U : x \notin A'\} = \{x \in U : x \in A\} = A$;

(iv) $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$;

(v) $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$.

EXERCISE 1.5

1. If A and B are two sets such that $A \subset B$, then find:
   (i) $A \cap B$ (ii) $A \cup B$
2. If $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{7, 8, 9, 10, 11\}$ and $D = \{10, 11, 12, 13, 14\}$, find:
   (i) $A \cup B$ (ii) $A \cup C$ (iii) $B \cup C$
   (iv) $B \cup D$ (v) $A \cup B \cup C$ (vi) $A \cup B \cup D$
   (vii) $B \cup C \cup D$ (viii) $A \cap (B \cup C)$ (ix) $(A \cap B) \cap (B \cap C)$
   (x) $(A \cup D) \cap (B \cup C)$.
3. Let $A = \{x : x \in N\}$, $B = \{x : x = 2n, n \in N\}$, $C = \{x : x = 2n - 1, n \in N\}$ and $D = \{x : x \text{ is a prime natural number}\}$. Find:
   (i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$
   (iv) $B \cap C$ (v) $B \cap D$ (vi) $C \cap D$
4. Let $A = \{3, 6, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$ and $D = \{5, 10, 15, 20\}$. Find:
   (i) $A - B$ (ii) $A - C$ (iii) $A - D$
   (iv) $B - A$ (v) $C - A$ (vi) $D - A$
   (vii) $B - C$ (viii) $B - D$
5. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find

(i) A' (ii) B' (iii) $(A \cap C)'$

(iv) $(A \cup B)'$ (v) $(A')'$ (vi) $(B - C)'$

6. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$.
Verify that (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

ANSWERS

1. (i) A (ii) B
2. (i) $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (ii) $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$
(iii) $\{4, 5, 6, 7, 8, 9, 10, 11\}$ (iv) $\{4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
(v) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ (vi) $\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}$
(vii) $\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ (viii) $\{4, 5\}$ (ix) ϕ (x) $\{4, 5, 10, 11\}$
3. (i) B (ii) C (iii) D (iv) ϕ (v) $\{2\}$ (vi) $D - \{2\}$
4. (i) $\{3, 6, 15, 18, 21\}$ (ii) $\{3, 15, 18, 21\}$ (iii) $\{3, 6, 12, 18, 21\}$
(iv) $\{4, 8, 16, 20\}$ (v) $\{2, 4, 8, 10, 14, 16\}$ (vi) $\{5, 10, 20\}$
(vii) $\{20\}$ (viii) $\{4, 8, 12, 16\}$
5. (i) $\{5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9\}$ (iii) $\{1, 2, 5, 6, 7, 8, 9\}$
(iv) $\{5, 7, 9\}$ (v) A (vi) $\{1, 3, 4, 5, 6, 7, 9\}$

1.9 LAWS OF ALGEBRA OF SETS

In this section, we shall state and prove some fundamental laws of algebra of sets.

THEOREM 1 (Idempotent Laws) *For any set A, we have*

(i) $A \cup A = A$ (ii) $A \cap A = A$.

PROOF (i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$.

THEOREM 2 (Identity Laws) *For any set A, we have*

(i) $A \cup \phi = A$ (ii) $A \cap U = A$.

i.e. ϕ and U are identity elements for union and intersection respectively.

PROOF (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

THEOREM 3 (Commutative Laws) *For any two sets A and B, we have*

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

PROOF Recall that two sets X and Y are equal iff $X \subseteq Y$ and $Y \subseteq X$. Also, $X \subseteq Y$ if every element of X belongs to Y.

(i) Let x be an arbitrary element of $A \cup B$. Then,

$x \in A \cup B \Rightarrow x \in A$ or $x \in B \Rightarrow x \in B$ or $x \in A \Rightarrow x \in B \cup A$

$\therefore$ $A \cup B \subseteq B \cup A$.

Similarly, $B \cup A \subseteq A \cup B$.

Hence, $A \cup B = B \cup A$.

THEOREM 4 (Associative Laws) *If A, B and C are any three sets, then*

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

i.e. union and intersection are associative.

PROOF (i) Let x be an arbitrary element of $(A \cup B) \cup C$. Then,

$x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\therefore$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Hence, $(A \cup B) \cup C = A \cup (B \cup C)$.

(ii) Let x be an arbitrary element of $A \cap (B \cap C)$. Then,

$x \in A \cap (B \cap C)$

$\Rightarrow$ $x \in A$ and $x \in (B \cap C)$

$\Rightarrow$ $x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow$ $x \in (A \cap B)$ and $x \in C$

$\Rightarrow$ $x \in (A \cap B) \cap C$

$\therefore$ $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

Similarly, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

Hence, $A \cap (B \cap C) = (A \cap B) \cap C$.

THEOREM 5 (Distributive Laws) *If A, B and C are any three sets, then*

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

i.e. union and intersection are distributive over intersection and union respectively.

PROOF (i) Let x be an arbitrary element of $A \cup (B \cap C)$. Then,

$x \in A \cup (B \cap C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cap C)$

$\Rightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$ [$\because$ 'or' is distributive over 'and']

$\Rightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow$ $x \in ((A \cup B) \cap (A \cup C))$

$\therefore$ $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) Let x be an arbitrary element of $A \cap (B \cup C)$. Then,

$x \in A \cap (B \cup C)$

$\Rightarrow$ $x \in A$ and $x \in (B \cup C)$

$\Rightarrow$ $x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow$ $x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

$\therefore$ $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

THEOREM 6 (De-morgan's Laws) *If A and B are any two sets, then*

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

PROOF (i) Let x be an arbitrary element of $(A \cup B)'$. Then,

$x \in (A \cup B)'$
$\Rightarrow x \notin (A \cup B)$
$\Rightarrow x \notin A$ and $x \notin B$
$\Rightarrow x \in A'$ and $x \in B'$
$\Rightarrow x \in A' \cap B'$.
$\therefore (A \cup B)' \subseteq A' \cap B'$.

Again, let y be an arbitrary element of $A' \cap B'$. Then,

$y \in A' \cap B'$
$\Rightarrow y \in A'$ and $y \in B'$
$\Rightarrow y \notin A$ and $y \notin B$
$\Rightarrow y \notin A \cup B$
$\Rightarrow y \in (A \cup B)'$
$\therefore A' \cap B' \subseteq (A \cup B)'$.

Hence, $(A \cup B)' = A' \cap B'$

(ii) Let x be an arbitrary element of $(A \cap B)'$. Then,

$x \in (A \cap B)'$
$\Rightarrow x \notin (A \cap B)$
$\Rightarrow x \notin A$ or $x \notin B$
$\Rightarrow x \in A'$ or $x \in B'$
$\Rightarrow x \in A' \cup B'$
$\Rightarrow (A \cap B)' \subseteq A' \cup B'$.

Again, let y be an arbitrary element of $A' \cup B'$. Then,

$y \in (A' \cup B')$
$\Rightarrow y \in A'$ or $y \in B'$
$\Rightarrow y \notin A$ or $y \notin B$
$\Rightarrow y \notin (A \cap B)$
$\Rightarrow y \in (A \cap B)'$
$\therefore A' \cup B' \subseteq (A \cap B)'$.

Hence, $(A \cap B)' = A' \cup B'$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *For any two sets A and B, prove that* $A \cup B = A \cap B \Leftrightarrow A = B$.

SOLUTION First let $A = B$. Then,

$A \cup B = A$ and $A \cap B = A$
$\Rightarrow A \cup B = A \cap B$

Thus, $A = B \Rightarrow (A \cup B) = (A \cap B)$ ...(i)

Conversely, let $A \cup B = A \cap B$. Then, we have to prove that $A = B$. For this, let

$x \in A$
$\Rightarrow x \in A \cup B$
$\Rightarrow x \in A \cap B$ [$\because A \cup B = A \cap B$]
$\Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in B$
$\therefore A \subset B$ ...(ii)

Now, let

$y \in B$

$\Rightarrow y \in A \cup B$
$\Rightarrow y \in A \cap B$ [$\because A \cup B = A \cap B$]
$\Rightarrow y \in A$ and $y \in B$
$\Rightarrow y \in A$
$\therefore B \subset A$ ...(iii)

From (ii) and (iii), we get $A = B$

Thus, $A \cup B = A \cap B \Rightarrow A = B$ ...(iv)

From (i) and (iv), we have

$A \cup B = A \cap B \Leftrightarrow A = B$

EXAMPLE 2 *Let A, B and C be the sets such that* $A \cup B = A \cup C$ *and* $A \cap B = A \cap C$. *Show that* $B = C$.

SOLUTION We have,

$A \cup B = A \cup C$
$\Rightarrow (A \cup B) \cap C = (A \cup C) \cap C$
$\Rightarrow (A \cap C) \cup (B \cap C) = C$ [$\because (A \cup C) \cap C = C$]
$\Rightarrow (A \cap B) \cup (B \cap C) = C$ [$\because A \cap C = A \cap B$] ...(i)

Again, $A \cup B = A \cup C$
$\Rightarrow (A \cup B) \cap B = (A \cup C) \cap B$
$\Rightarrow B = (A \cap B) \cup (C \cap B)$ [$\because (A \cup B) \cap B = B$]
$\Rightarrow B = (A \cap B) \cup (B \cap C)$ ...(ii)

From (i) and (ii), we get

$B = C$

EXAMPLE 3 *Let A and B be sets, if* $A \cap X = B \cap X = \phi$ *and* $A \cup X = B \cup X$ *for some set X, prove that* $A = B$.

SOLUTION We have,

$A \cup X = B \cup X$ for some set X
$\Rightarrow A \cap (A \cup X) = A \cap (B \cup X)$
$\Rightarrow A = (A \cap B) \cup (A \cap X)$ [$\because A \cap (A \cup X) = A$]
$\Rightarrow A = (A \cap B) \cup \phi$ [$\because A \cap X = \phi$ (given)]
$\Rightarrow A = A \cap B$
$\Rightarrow A \subset B$ ...(i)

Again, $A \cup X = B \cup X$
$\Rightarrow B \cap (A \cup X) = B \cap (B \cup X)$
$\Rightarrow (B \cap A) \cup (B \cap X) = B$ [$\because B \cap (B \cup X) = B$]
$\Rightarrow (B \cap A) \cup \phi = B$ [$\because B \cap X = \phi$ (given)]
$\Rightarrow B \cap A = B$
$\Rightarrow A \cap B = B$
$\Rightarrow B \subset A$ ...(ii)

From (i) and (ii), we get $A = B$.

EXAMPLE 4 *For any two sets A and B, prove that:* $P(A) = P(B) \Rightarrow A = B$

SOLUTION Let x be an arbitrary element of A. Then, there exists a subset, say X, of set A such that $x \in X$.

Now, $X \subset A$

$\Rightarrow X \in P(A)$

$\Rightarrow X \in P(B)$ $\quad [\because P(A) = P(B)]$

$\Rightarrow X \subset B$

$\Rightarrow x \in B$ $\quad [\because x \in X \text{ and } X \subset B \therefore x \in B]$

Thus, $x \in A \Rightarrow x \in B$

$\therefore A \subset B$ ...(i)

Now, let y be an arbitrary element of B. Then, there exists a subset, say Y, of set B such that $y \in Y$.

Now, $Y \subset B$

$\Rightarrow Y \in P(B)$

$\Rightarrow Y \in P(A)$ $\quad [\because P(A) = P(B)]$

$\Rightarrow Y \subset A$

$\Rightarrow y \in A$

Thus, $y \in B \Rightarrow y \in A$

$\therefore B \subset A$ ...(ii)

From (i) and (ii), we obtain $A = B$.

EXAMPLE 5 *For any two sets A and B prove that:* $P(A \cap B) = P(A) \cap P(B)$.

SOLUTION In order to prove that $P(A \cap B) = P(A) \cap P(B)$, it is sufficient to prove that

$P(A \cap B) \subset P(A) \cap P(B)$ and $P(A) \cap P(B) \subset P(A \cap B)$.

First let

$X \in P(A \cap B)$

$\Rightarrow X \subset A \cap B$

$\Rightarrow X \subset A$ and $X \subset B$

$\Rightarrow X \in P(A)$ and $X \in P(B)$

$\Rightarrow X \in P(A) \cap P(B)$

$\therefore P(A \cap B) \subset P(A) \cap P(B)$ ...(i)

Now, let

$Y \in P(A) \cap P(B)$. Then,

$Y \in P(A) \cap P(B)$

$\Rightarrow Y \in P(A)$ and $Y \in P(B)$

$\Rightarrow Y \subset A$ and $Y \subset B$

$\Rightarrow Y \subset A \cap B$

$\Rightarrow Y \in P(A \cap B)$

$\therefore P(A) \cap P(B) \subset P(A \cap B)$ ...(ii)

From (i) and (ii), we get

$P(A \cap B) = P(A) \cap P(B)$.

EXAMPLE 6 *For any two sets A and B prove that* $P(A) \cup P(B) \subset P(A \cup B)$. *But,* $P(A \cup B)$ *is not necessarily a subset of* $P(A) \cup P(B)$.

SOLUTION Let $X \in P(A) \cup P(B)$. Then,

$X \in P(A) \cup P(B)$

$\Rightarrow X \in P(A)$ or $X \in P(B)$

$\Rightarrow X \subset A$ or $X \subset B$

$\Rightarrow X \subset A \cup B$

$\Rightarrow X \in P(A \cup B)$

$\therefore P(A) \cup P(B) \subset P(A \cup B)$

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Then, we find that $X = \{1, 2, 3, 4\} \subset (A \cup B)$. Therefore, $X \in P(A \cup B)$. But, $X \notin P(A)$, $X \notin P(B)$. So, $X \notin P(A) \cup P(B)$.

Thus, $P(A \cup B)$ is not necessarily a subset of $P(A) \cup P(B)$.

EXAMPLE 7 *If* $a \in N$ *such that* $aN = \{ax : x \in N\}$. *Describe the set* $3N \cap 7N$.

SOLUTION We have, $aN = \{ax : x \in N\}$

$\therefore 3N = \{3x : x \in N\} = \{3, 6, 9, 12, \ldots\}$ and, $7N = \{7x : x \in N\} = \{7, 14, 21, 28, \ldots\}$

Hence, $3N \cap 7N = \{21, 42, \ldots\} = \{21x : x \in N\} = 21N$.

EXAMPLE 8 *If* $A = \{1, 3, 5, 7, 11, 13, 15, 17\}$, $B = \{2, 4, 6, \ldots, 18\}$ *and N is the universal set, then find* $A' \cup ((A \cup B) \cap B')$.

SOLUTION We have, $(A \cup B) \cap B' = A$ $\quad [\because A, B \text{ are disjoint sets}]$

$\therefore A' \cup ((A \cup B) \cap B') = A' \cup A = N$.

EXAMPLE 9 *For any natural number a, we define* $aN = \{ax : x \in N\}$. *If* $b, c, d \in N$ *such that* $bN \cap cN = dN$, *then prove that d is the l.c.m. of b and c.*

SOLUTION We have,

$bN = \{bx : x \in N\}$ = the set of positive integral multiples of b

$cN = \{cx : x \in N\}$ = the set of positive integral multiples of c

$\therefore bN \cap cN$ = the set of positive integral multiples of b and c both.

$\Rightarrow bN \cap cN = \{kx : x \in N\}$, where k is the l.c.m. of b and c.

Hence, d = l.c.m. of b and c.

EXAMPLE 10 *Suppose* $A_1, A_2, \ldots, A_{30}$ *are thirty sets each with five elements and* $B_1, B_2, \ldots, B_n$ *are n sets each with three elements. Let* $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. *Assume that each element of S belongs to exactly ten of the* A_i*'s and exactly 9 of* B_j*'s. Find n.*

SOLUTION Since each A_i has 5 elements and each element of S belongs to exactly 10 of A_i's.

$$\therefore \quad S = \bigcup_{i=1}^{30} A_i \Rightarrow n(S) = \frac{1}{10} \sum_{i=1}^{30} n(A_i) = \frac{1}{10}(5 \times 30) = 15 \quad \text{...(i)}$$

Again, each B_j has 3 elements and each element of S belongs to exactly 9 of B_j's

$$\therefore \quad S = \bigcup_{j=1}^{n} B_j \Rightarrow n(S) = \frac{1}{9} \sum_{j=1}^{n} n(B_j) = \frac{1}{9}(3n) = \frac{n}{3} \quad \text{...(ii)}$$

From (i) and (ii), we get : $15 = \frac{n}{3} \Rightarrow n = 45$.

EXERCISE 1.6

1. Find the smallest set A such that $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$.
2. Let $A = \{1, 2, 4, 5\}$ $B = \{2, 3, 5, 6\}$ $C = \{4, 5, 6, 7\}$. Verify the following identities:
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$
   (iv) $A - (B \cup C) = (A - B) \cap (A - C)$
   (v) $A - (B \cap C) = (A - B) \cup (A - C)$
   (vi) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.
3. If $U = \{2, 3, 5, 7, 9\}$ is the universal set and $A = \{3, 7\}$, $B = \{2, 5, 7, 9\}$, then prove that:
   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.
4. For any two sets A and B, show that the following statements are equivalent:
   (i) $A \subset B$ (ii) $A - B = \phi$ (iii) $A \cup B = B$
   (iv) $A \cap B = A$.
5. For three sets A, B and C, show that
   (i) $A \cap B = A \cap C$ need not imply $B = C$.
   (ii) $A \subset B \Rightarrow C - B \subset C - A$
6. For any two sets, prove that:
   (i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$
7. Find sets A, B and C such that $A \cap B$, $A \cap C$ and $B \cap C$ are non-empty sets and $A \cap B \cap C = \phi$.
8. For any two sets A and B, prove that : $A \cap B = \phi \Rightarrow A \subseteq B'$.
9. If A and B are sets, then prove that $A - B$, $A \cap B$ and $B - A$ are pairwise disjoint.
10. Using properties of sets, show that for any two sets A and B,
$$(A \cup B) \cap (A \cap B') = A$$
11. For any two sets of A and B, prove that:
   (i) $A' \cup B = U \Rightarrow A \subset B$
   (ii) $B' \subset A' \Rightarrow A \subset B$
12. Is it true that for any sets A and B, $P(A) \cup P(B) = P(A \cup B)$? Justify your answer.
13. Show that for any sets A and B,
   (i) $A = (A \cap B) \cup (A - B)$ (ii) $A \cup (B - A) = A \cup B$

ANSWERS

1. $A = \{3, 5, 9\}$ 7. $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$ 12. False

HINTS TO SELECTED PROBLEMS

4. (i) $\Rightarrow$ (ii)
   We ahve, $A - B = \{x \in A : x \notin B\}$
   Since $A \subset B$. Therefore, there is no element in A which does not belong to B.
   $\therefore$ $A - B = \phi$
   Hence, (i) $\Rightarrow$ (ii).
   (ii) $\Rightarrow$ (iii)
   We have,
   $$A - B = \phi \Rightarrow A \subset B \Rightarrow A \cup B = B$$
   Hence, (ii) $\Rightarrow$ (iii).

   (iii) $\Rightarrow$ (iv)
   We have, $A \cup B = B \Rightarrow A \subset B \Rightarrow A \cap B = A$
   $\therefore$ (iii) $\Rightarrow$ (iv)
   (iv) $\Rightarrow$ (i)
   We have, $A \cap B = A \Rightarrow A \subset B$
   $\therefore$ (iv) $\Rightarrow$ (i)
   Hence, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)
5. (i) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{1, 3, 4, 7, 8\}$. Then, $A \cap B = A \cap C$, but $B \neq C$
   (ii) Let $x \in C - B$. Then,
   $x \in C - B$
   $\Rightarrow$ $x \in C$ and $x \notin B$
   $\Rightarrow$ $x \in C$ and $x \notin A$ $[\because A \subset B]$
   $\Rightarrow$ $x \in C - A$
   $\therefore$ $C - B \subset C - A$
6. (i) We have,
   $A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$ [$\because$ U is distributive over $\cap$]
   $= A \cap (A \cup B)$
   $= A$ $[\because A \subset A \cup B]$
   (ii) We have,
   $A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$ [$\because$ $\cap$ is distributive over U]
   $= A \cup (A \cap B)$
   $= A$ $[\because A \cap B \subset A]$
7. $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$
8. $x \in A \Rightarrow x \notin B$ $[\because A \cap B = \phi]$
   $\Rightarrow x \in B'$.
   So, $A \subseteq B'$
9. We have,
   $A - B = \{x : x \in A \text{ and } x \notin B\}$, $B - A = \{x \in B \text{ and } x \notin A\}$
   $\therefore$ $A - B$ and $B - A$ are disjoint sets
   Now,
   $x \in A - B \Rightarrow x \in A$ and $x \notin B \Rightarrow x \notin A \cap B$
   $\therefore$ $(A - B)$ and $A \cap B$ are disjoint sets.
   Similarly, $B - A$ and $A \cap B$ are disjoint sets.
10. $(A \cup B) \cap (A \cup B')$
   $= ((A \cup B) \cap A) \cup ((A \cup B) \cap B')$
   $= A \cup ((A \cup B) \cap B') = A \cup ((A \cap B') \cup (B \cap B')) = A \cup (A \cap B') = A$
11. (i) Let $x \in A$. Then,
   $x \in A \Rightarrow x \in U \Rightarrow x \in A' \cup B \Rightarrow x \in B$ $[\because x \notin A']$
   $\therefore$ $A \subset B$
   (ii) Let $x \in A$. Then,
   $x \in A \Rightarrow x \notin A' \Rightarrow x \notin B' \Rightarrow x \in B$
   $\therefore$ $A \subset B$ $[\because B' \subset A']$

1.10 MORE RESULTS ON OPERATIONS ON SETS

THEOREM 1 *If A and B are any two sets, then*

(i) $A - B = A \cap B'$ (ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$ (vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$

(vii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

PROOF (i) Let x be an arbitrary element of $A - B$. Then,

$x \in (A - B)$

$\Rightarrow$ $x \in A$ and $x \notin B$

$\Rightarrow$ $x \in A$ and $x \in B'$

$\Rightarrow$ $x \in A \cap B'$

$\therefore$ $A - B \subseteq A \cap B'$ ...(i)

Again, let y be an arbitrary element of $A \cap B'$. Then,

$y \in A \cap B'$

$\Rightarrow$ $y \in A$ and $y \in B'$

$\Rightarrow$ $y \in A$ and $y \notin B$

$\Rightarrow$ $y \in A - B$

$\therefore$ $A \cap B' \subseteq (A - B)$ ...(ii)

Hence, from (i) and (ii), we have $A - B = A \cap B'$.

(ii) Proceed as in (i).

(iii) In order to prove that $A - B = A \Leftrightarrow A \cap B = \phi$,

we shall prove that: (i) $A - B = A \Rightarrow A \cap B = \phi$. (ii) $A \cap B = \phi \Rightarrow A - B = A$.

First, let $A - B = A$. Then we have to prove that $A \cap B = \phi$. If possible, let $A \cap B \neq \phi$. Then,

$A \cap B \neq \phi$

$\Rightarrow$ there exists $x \in A \cap B$

$\Rightarrow$ $x \in A$ and $x \in B \Rightarrow x \in A - B$ and $x \in B$ [$\because A - B = A$]

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \in B$ [By def. of $A - B$]

$\Rightarrow$ $x \in A$ and ($x \notin B$ and $x \in B$)

But $x \notin B$ and $x \in B$ both can never be possible simultaneously. Thus, we arrive at a contradiction. So, our supposition is wrong.

$\therefore$ $A \cap B = \phi$

Hence, $A - B = A \Rightarrow A \cap B = \phi$ ...(i)

Conversely, let $A \cap B = \phi$. Then we have to prove that $A - B = A$. For this we shall show that $A - B \subseteq A$ and $A \subseteq A - B$.

Let x be an arbitrary element of $A - B$. Then,

$x \in A - B \Rightarrow x \in A$ and $x \notin B \Rightarrow x \in A$

$\therefore$ $A - B \subseteq A$.

Again let y be an arbitrary element of A. Then,

$y \in A$

$\Rightarrow$ $y \in A$ and $y \notin B$ [$\because A \cap B = \phi$]

$\Rightarrow$ $y \in A - B$ [By def. of $A - B$]

$\therefore$ $A \subseteq A - B$.

So, we have $A - B \subseteq A$ and $A \subseteq A - B$. Therefore, $A - B = A$.

Thus, $A \cap B = \phi \Rightarrow A - B = A$ ...(ii)

Hence, from (i) and (ii), we have

$A - B = A \Leftrightarrow A \cap B = \phi$.

(iv) Let x be an arbitrary element of $(A - B) \cup B$. Then,

$x \in (A - B) \cup B$

$\Rightarrow$ $x \in A - B$ or $x \in B$

$\Rightarrow$ ($x \in A$ and $x \notin B$) or $x \in B$

$\Rightarrow$ ($x \in A$ or $x \in B$) and ($x \notin B$ or $x \in B$)

$\Rightarrow$ $x \in A \cup B$

$\therefore$ $(A - B) \cup B \subseteq A \cup B$

Let y be an arbitrary element of $A \cup B$. Then,

$y \in A \cup B$

$\Rightarrow$ $y \in A$ or $y \in B$

$\Rightarrow$ ($y \in A$ or $y \in B$) and ($y \notin B$ or $y \in B$)

$\Rightarrow$ ($y \in A$ and $y \notin B$) or $y \in B$

$\Rightarrow$ $y \in (A - B) \cup B$

$\therefore$ $A \cup B \subseteq (A - B) \cup B$

Hence, $(A - B) \cup B = A \cup B$.

(v) If possible let $(A - B) \cap B \neq \phi$. Then, there exists at least one element x, (say), in $(A - B) \cap B$.

Now, $x \in (A - B) \cap B$

$\Rightarrow$ $x \in (A - B)$ and $x \in B$

$\Rightarrow$ ($x \in A$ and $x \notin B$) and $x \in B$

$\Rightarrow$ $x \in A$ and ($x \notin B$ and $x \in B$)

But, $x \notin B$ and $x \in B$ both can never be possible simultaneously. Thus, we arrive at a contradiction. So, our supposition is wrong.

Hence, $(A - B) \cap B = \phi$

(vi) First, let $A \subseteq B$. Then we have to prove that $B' \subseteq A'$. Let x be an arbitrary element of B'. Then,

$x \in B'$

$\Rightarrow$ $x \notin B$

$\Rightarrow$ $x \notin A$ [$\because A \subseteq B$]

$\Rightarrow$ $x \in A'$

$\therefore$ $B' \subseteq A'$.

Thus, $A \subseteq B \Rightarrow B' \subseteq A'$ ...(i)

Conversely, let $B' \subseteq A'$. Then, we have to prove that $A \subseteq B$. Let y be an arbitrary element of A. Then,

$y \in A$

$\Rightarrow$ $y \notin A'$

$\Rightarrow$ $y \notin B'$ [$\because B' \subseteq A'$]

$\Rightarrow$ $y \in B$

$\therefore$ $A \subseteq B$.

Thus, $B' \subseteq A' \Rightarrow A \subseteq B$

From (i) and (ii), we have $A \subseteq B \Leftrightarrow B' \subseteq A'$. ...(ii)

(vii) Let x be an arbitrary element of $(A-B)\cup(B-A)$. Then,

$x \in (A-B)\cup(B-A)$

$\Rightarrow$ $x \in A-B$ or $x \in B-A$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow$ $(x \in A$ or $x \in B)$ and $(x \notin B$ or $x \notin A)$

$\Rightarrow$ $x \in (A\cup B)$ and $x \notin (A\cap B)$

$\Rightarrow$ $x \in (A\cup B)-(A\cap B)$

$\therefore$ $(A-B)\cup(B-A) \subseteq (A\cup B)-(A\cap B)$ ...(i)

Again, let y be an arbitrary element of $(A\cup B)-(A\cap B)$.

Then, $y \in (A\cup B)-(A\cap B)$

$\Rightarrow$ $y \in A\cup B$ and $y \notin A\cap B$

$\Rightarrow$ $(y \in A$ or $y \in B)$ and $(y \notin A$ or $y \notin B)$

$\Rightarrow$ $(y \in A$ and $y \notin B)$ or $(y \in B$ and $y \notin A)$

$\Rightarrow$ $y \in (A-B)$ or $y \in (B-A)$ $\Rightarrow$ $y \in (A-B)\cup(B-A)$.

$\therefore$ $(A\cup B)-(A\cap B) \subseteq (A-B)\cup(B-A)$ ...(ii)

Hence, from (i) and (ii), we have

$(A-B)\cup(B-A) = (A\cup B)-(A\cap B)$.

THEOREM 2 *If A, B and C are any three sets, then prove that:*

(i) $A-(B\cap C)=(A-B)\cup(A-C)$

(ii) $A-(B\cup C)=(A-B)\cap(A-C)$

(iii) $A\cap(B-C)=(A\cap B)-(A\cap C)$

(iv) $A\cap(B\Delta C)=(A\cap B)\Delta(A\cap C)$

PROOF (i) Let x be any element of $A-(B\cap C)$. Then,

$x \in A-(B\cap C)$

$\Rightarrow$ $x \in A$ and $x \notin (B\cap C)$

$\Rightarrow$ $x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$

$\Rightarrow$ $x \in (A-B)$ or $x \in (A-C)$

$\Rightarrow$ $x \in (A-B)\cup(A-C)$

$\therefore$ $A-(B\cap C) \subseteq (A-B)\cup(A-C)$

Similarly, $(A-B)\cup(A-C) \subseteq A-(B\cap C)$

Hence, $A-(B\cap C)=(A-B)\cup(A-C)$

(ii) Let x be an arbitrary element of $A-(B\cup C)$. Then

$x \in A-(B\cup C)$

$\Rightarrow$ $x \in A$ and $x \notin (B\cup C)$

$\Rightarrow$ $x \in A$ and $(x \notin B$ and $x \notin C)$

$\Rightarrow$ $(x \in A$ and $x \notin B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow$ $x \in (A-B)$ and $x \in A-C$

$\Rightarrow$ $x \in (A-B)\cap(A-C)$

$\therefore$ $A-(B\cup C) \subseteq (A-B)\cap(A-C)$

Similarly, $(A-B)\cap(A-C) \subseteq A-(B\cup C)$

Hence, $A-(B\cup C)=(A-B)\cap(A-C)$

(iii) Let x be any arbitrary element of $A\cap(B-C)$. Then

$x \in A\cap(B-C)$

$\Rightarrow$ $x \in A$ and $x \in (B-C)$

$\Rightarrow$ $x \in A$ and $(x \in B$ and $x \notin C)$

$\Rightarrow$ $(x \in A$ and $x \in B)$ and $(x \in A$ and $x \notin C)$

$\Rightarrow$ $x \in (A\cap B)$ and $x \notin (A\cap C)$

$\Rightarrow$ $x \in (A\cap B)-(A\cap C)$

$\therefore$ $A\cap(B-C) \subseteq (A\cap B)-(A\cap C)$

Similarly, $(A\cap B)-(A\cap C) \subseteq A\cap(B-C)$

Hence, $A\cap(B-C)=(A\cap B)-(A\cap C)$.

(iv) $A\cap(B\Delta C) = A\cap[(B-C)\cup(C-B)]$

$= [A\cap(B-C)]\cup[A\cap(C-B)]$ [By distributive law]

$= [(A\cap B)-(A\cap C)]\cup[(A\cap C)-(A\cap B)]$ [Using (iii)]

$= (A\cap B)\Delta(A\cap C)$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Let A and B be two sets. Using properties of sets prove that:*

(i) $A\cap B' = \phi \Rightarrow A\subset B$ (ii) $A'\cup B = U \Rightarrow A\subset B$

SOLUTION

(i) We have,

$A = (A\cap U)$

$\Rightarrow$ $A = A\cap(B\cup B')$

$\Rightarrow$ $A = (A\cap B)\cup(A\cap B')$ [$\because$ $\cap$ is distributive over union]

$\Rightarrow$ $A = (A\cap B)\cup\phi$ [$\because A\cap B' = \phi$]

$\Rightarrow$ $A = A\cap B$

$\therefore$ $A\subset B$

(ii) From (i), we have

$A\cap B' = \phi$

$\Leftrightarrow$ $(A\cap B')' = \phi'$

$\Leftrightarrow$ $A'\cup(B')' = U$ [$\because \phi' = U$]

$\Leftrightarrow$ $A'\cup B = U$ [$\because (B')' = B$]

Thus, $A\cap B' = \phi \Leftrightarrow A'\cup B = U$

and $A\cap B' = \phi \Rightarrow A\subset B$

$\therefore$ $A'\cup B = U \Rightarrow A\subset B$

ALITER We have,

$A'\cup B = U$

$\Rightarrow$ $A\cap(A'\cup B) = A\cap U$ [Taking intersection with A]

$\Rightarrow$ $(A\cap A')\cup(A\cap B) = A$ [$\because A\cap U = A$]

$\Rightarrow$ $\phi\cup(A\cap B) = A$

$\Rightarrow$ $A\cap B = A$

$\Rightarrow$ $A\subset B$

EXAMPLE 2 *Let A and B be two sets. Prove that:*

$(A-B)\cup B = A$ *if and only if* $B\subset A$

SOLUTION First let, $(A-B)\cup B=A$. Then, we have to prove that $B\subset A$.

Now, $(A-B)\cup B=A$

$\Rightarrow (A\cap B')\cup B=A$ $[\because A-B=A\cap B']$

$\Rightarrow (A\cup B)\cap(B'\cup B)=A$

$\Rightarrow (A\cup B)\cap U=A$

$\Rightarrow A\cup B=A$

$\Rightarrow B\subset A$

Conversely, let $B\subset A$. Then, we have to prove that $(A-B)\cup B=A$.

Now, $(A-B)\cup B=(A\cap B')\cup B$

$\Rightarrow (A-B)\cup B=(A\cup B)\cap(B'\cup B)$

$\Rightarrow (A-B)\cup B=(A\cup B)\cap U$

$\Rightarrow (A-B)\cup B=A\cup B$

$\Rightarrow (A-B)\cup B=A$ $[\because B\subset A \therefore A\cup B=A]$

EXAMPLE 3 *Let A, B and C be three sets such that $A\cup B=C$ and $A\cap B=\phi$. Then, prove that $A=C-B$.*

SOLUTION We have, $A\cup B=C$.

$\therefore$ $C-B=(A\cup B)-B$

$=(A\cup B)\cap B'$ $[\because X-Y=X\cap Y']$

$=(A\cap B')\cup(B\cap B')$

$=(A\cap B')\cup\phi$

$=A\cap B'$

$=A-B$

$=A$ $[\because A\cap B=\phi]$

EXAMPLE 4 *Let A and B be any two sets. Using properties of sets prove that :*

(i) $(A-B)\cup B=A\cup B$ (ii) $(A-B)\cup A=A$

(iii) $(A-B)\cap B=\phi$ (iv) $(A-B)\cap A=A\cap B'$

SOLUTION (i) We have,

$(A-B)\cup B=(A\cap B')\cup B$ $[\because A-B=A\cap B']$

$=(A\cup B)\cap(B'\cup B)$ $[\because U$ is distributive over $\cap]$

$=(A\cup B)\cap U$ $[\because B'\cup B=U]$

$=A\cup B$

(ii) $(A-B)\cup A=A$ $[\because A-B\subset A]$

(iii) $(A-B)\cap B=(A\cap B')\cap B$

$=A\cap(B'\cap B)$

$=A\cap\phi$

$=\phi$

(iv) $(A-B)\cap A=A-B$ $[\because A-B\subset A]$

$=A\cap B'$

EXAMPLE 5 *For any two sets A and B prove by using properties of sets that :*

(i) $(A\cup B)-(A\cap B)=(A-B)\cup(B-A)$

(ii) $(A\cap B)\cup(A-B)=A$

(iii) $(A\cup B)-A=B-A$

SOLUTION (i) We have,

$(A\cup B)-(A\cap B)=(A\cup B)\cap(A\cap B)'$ $[\because X-Y=X\cap Y']$

$=(A\cup B)\cap(A'\cup B')$ $[\because (A\cap B)'=A'\cup B']$

$=X\cap(A'\cup B')$, where $X=A\cup B$

$=(X\cap A')\cup(X\cap B')$

$=(B\cap A')\cup(A\cap B')$ $\left[\begin{array}{l}\because X\cap A'=(A\cup B)\cap A'\\ =(A\cap A')\cup(B\cap A')=\phi\cup(B\cap A')\\ =B\cap A' \text{ Similarly, } X\cap B'=A\cap B'\end{array}\right]$

$=(A\cap B')\cup(B\cap A')$

$=(A-B)\cup(B-A)$ $[\because A-B=A\cap B'$ and $B-A=B\cap A']$

(ii) $(A\cap B)\cup(A-B)=(A\cap B)\cup(A\cap B')$

$=X\cup(A\cap B')$, where $X=A\cap B$

$=(X\cup A)\cap(X\cup B')$

$=A\cap(A\cup B')$ $\left[\begin{array}{l}\because X\cup A=(A\cap B)\cup A=A\ [\because A\cap B\subset A]\\ X\cup B'=(A\cap B)\cup B'=(A\cup B')\cap(B\cup B')\\ =(A\cup B')\cap U=A\cup B'\end{array}\right]$

$=A$ $[\because A\subset A\cup B']$

(iii) $(A\cup B)-A=(A\cup B)\cap A'$ $[\because X-Y=X\cap Y']$

$=(A\cap A')\cup(B\cap A')$

$=\phi\cup(B\cap A')$ $[\because A\cap A'=\phi]$

$=B\cap A'$

$=B-A$ $[\because B-A=B\cap A']$

EXAMPLE 6 *For sets A, B and C using properties of sets, prove that :*

(i) $A-(B\cup C)=(A-B)\cap(A-C)$

(ii) $A-(B\cap C)=(A-B)\cup(A-C)$

(iii) $(A\cup B)-C=(A-C)\cup(B-C)$

SOLUTION (i) We have,

$A-(B\cup C)=A\cap(B\cup C)'$ $[\because X-Y=X\cap Y']$

$=A\cap(B'\cap C')$ $[\because (B\cup C)'=B'\cap C']$

$=(A\cap B')\cap(A\cap C')$

$=(A-B)\cap(A-C)$

(ii) $A-(B\cap C)=A\cap(B\cap C)'$ $[\because X-Y=X\cap Y']$

$=A\cap(B'\cup C')$ $[\because (B\cap C)'=B'\cup C']$

$=(A\cap B')\cup(A\cap C')$ $[\because \cap$ is distributive over $U]$

$=(A-B)\cup(A-C)$

(iii) $(A\cup B)-C=(A\cup B)\cap C'$ $[\because X-Y=X\cap Y']$

$=(A\cap C')\cup(B\cap C')$

$=(A-C)\cup(B-C)$

EXAMPLE 7 *For sets A, B and C using properties of sets, prove that :*

(i) $A-(B-C)=(A-B)\cup(A\cap C)$ (ii) $A\cap(B-C)=(A\cap B)-(A\cap C)$

SOLUTION (i) We have,

$$A-(B-C)=A-(B\cap C') \qquad [\because B-C=B\cap C']$$
$$=A\cap(B\cap C')' \qquad [\because X-Y=X\cap Y']$$
$$=A\cap(B'\cup C) \qquad [\because (B\cap C')'=B'\cup(C')'=B'\cup C]$$
$$=(A\cap B')\cup(A\cap C)$$
$$=(A-B)\cup(A\cap C)$$

(ii)
$$A\cap(B-C)=A\cap(B\cap C') \qquad [\because B-C=B\cap C']$$
$$=(A\cap B)\cap C'$$
$$=\phi\cup((A\cap B)\cap C')$$
$$=((A\cap B)\cap A')\cup((A\cap B)\cap C') \qquad [\because (A\cap B)\cap A'=\phi]$$
$$=(A\cap B)\cap(A'\cup C')$$
$$=(A\cap B)\cap(A\cap C)'$$
$$=(A\cap B)-(A\cap C)$$

EXERCISE 1.7

1. For any two sets A and B, prove that : $A'-B'=B-A$
2. For any two sets A and B, prove the following :
   (i) $A\cap(A'\cup B)=A\cap B$ (ii) $A-(A-B)=A\cap B$
   (iii) $A\cap(A\cup B)'=\phi$ (iv) $A-B=A\Delta(A\cap B)$.
3. If A, B, C are three sets such that $A\subset B$, then prove that
   $C-B\subset C-A$.

HINTS TO SELECTED PROBLEMS

1. We know that $X-Y=X\cap Y'$. So $A'-B'=A'\cap(B')'=A'\cap B=B\cap A'=B-A$
2. (ii) $A-(A-B)=A-(A\cap B')=A\cap(A\cap B')'=A\cap(A'\cup(B')')$
   $=A\cap(A'\cup(B')')=A\cap B$
3. Let $x\in C-B$. Then,

$$x\in C-B$$
$$\Rightarrow x\in C \text{ and } x\notin B$$
$$\Rightarrow x\in C \text{ and } x\notin A$$
$$\Rightarrow x\in C-A \qquad [\because A\subset B]$$
$$\therefore C-B\subset C-A.$$

1.11 SOME IMPORTANT RESULTS ON NUMBER OF ELEMENTS IN SETS

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

(ii) $n(A\cup B)=n(A)+n(B) \Leftrightarrow A, B$ are disjoint non-void sets.

(iii) $n(A-B)=n(A)-n(A\cap B)$ i.e. $n(A-B)+n(A\cap B)=n(A)$

(iv) $n(A\Delta B)=$ No. of elements which belong to exactly one of A or B

$$=n((A-B)\cup(B-A))$$
$$=n(A-B)+n(B-A) \qquad [\because (A-B) \text{ and } (B-A) \text{ are disjoint}]$$
$$=n(A)-n(A\cap B)+n(B)-n(A\cap B)$$
$$=n(A)+n(B)-2n(A\cap B)$$

(v) $n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)$
$\quad -n(A\cap C)+n(A\cap B\cap C)$

(vi) No. of elements in exactly two of the sets A, B, C
$$=n(A\cap B)+n(B\cap C)+n(C\cap A)-3n(A\cap B\cap C)$$

(vii) No. of elements in exactly one of the sets A, B, C
$$=n(A)+n(B)+n(C)-2n(A\cap B)-2n(B\cap C)-2n(A\cap C)+3n(A\cap B\cap C)$$

(viii) $n(A'\cup B')=n((A\cap B)')=n(U)-n(A\cap B)$

(ix) $n(A'\cap B')=n((A\cup B)')=n(U)-n(A\cup B)$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If X and Y are two sets such that $n(X)=17$, $n(Y)=23$ and $n(X\cup Y)=38$, find $n(X\cap Y)$.*

SOLUTION We have,

$$n(X\cup Y)=n(X)+n(Y)-n(X\cap Y)$$
$$\Rightarrow 38=17+23-n(X\cap Y)$$
$$\Rightarrow n(X\cap Y)=40-38=2.$$

EXAMPLE 2 *In a group of 800 people, 550 can speak Hindi and 450 can speak English. How many can speak both Hindi and English ?*

SOLUTION Let H denote the set of people speaking Hindi and E denote the set of people speaking English. We are given that

$$n(H)=550, n(E)=450 \text{ and } n(H\cup E)=800.$$

We have to find $n(H\cap E)$.

We know that

$$n(H\cup E)=n(H)+n(E)-n(H\cap E)$$
$$\Rightarrow n(H\cap E)=n(H)+n(E)-n(H\cup E)$$
$$\Rightarrow n(H\cap E)=550+450-800=200.$$

Hence, 200 persons can speak both Hindi and English.

EXAMPLE 3 *In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football?*

SOLUTION Let C be the set of students who like to play cricket and F be the set of students who like to play football. Then, $C\cup F$ is the set of students who like to play at least one game and, $C\cap F$ is the set of all students who like to play both games.

We have,

$$n(C)=24, n(F)=16, n(C\cup F)=35 \text{ and we have to find } n(C\cap F).$$

Now,

$$n(C\cup F)=n(C)+n(F)-n(C\cap F)$$
$$\Rightarrow 35=24+16-n(C\cap F)$$
$$\Rightarrow n(C\cap F)=40-35=5$$

EXAMPLE 4 *In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all the people speak at least one of the two languages. How many people speak only English and not Hindi ? How many people speak English?*

SOLUTION Let H denote the set of people speaking Hindi and E the set of people speaking English. Then, it is given that

$$n(H\cup E)=50, n(H)=35, n(H\cap E)=25.$$

Now, $n(E-H) = n(H \cup E) - n(H) = 50 - 35 = 15$

Thus, the number of people speaking English but not Hindi is 15.

We have, $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\Rightarrow \quad 50 = 35 + n(E) - 25 \Rightarrow n(E) = 40.$

Hence, the number of people who speak English is 40.

EXAMPLE 5 *There are 200 individuals with a skin disorder, 120 had been exposed to chemical C_1, 50 to chemical C_2, and 30 to both the chemicals C_1 and C_2. Find the number of individuals exposed to (i) chemical C_1 or chemical C_2 (ii) chemical C_1 but not chemical C_2 (iii) chemical C_2 but not chemical C_1.*

SOLUTION Let U denote the universal set consisting of individuals suffering from the skin disorder, A denote the set of individuals exposed to chemical C_1 and B denote the set of individuals exposed to chemical C_2.

We have,

$n(U) = 200, n(A) = 120, n(B) = 50$ and $n(A \cap B) = 30.$

(i) Number of individuals exposed to chemical C_1 or chemical C_2

$= n(A \cup B)$

$= n(A) + n(B) - n(A \cap B)$

$= 120 + 50 - 30 = 140$

(ii) Number of individuals exposed to chemical C_1 but not chemical C_2

$= n(A \cap \overline{B})$

$= n(A) - n(A \cap B)$

$= 120 - 30 = 90.$

(iii) Number of individuals exposed to chemical C_2 but not chemical C_1

$= n(\overline{A} \cap B)$

$= n(B) - n(A \cap B)$

$= 50 - 30 = 20.$

EXAMPLE 6 *Out of 500 car owners investigated, 400 owned Maruti car and 200 owned Hyundai car; 50 owned both cars. Is this data correct?*

SOLUTION Let U be the set of all car owners investigated, M be the set of persons who owned Maruti cars and H be the set of persons who owned Hyundai cars.

It is given that

$n(U) = 500, n(M) = 400, n(H) = 200$ and $n(M \cap H) = 50.$

$\therefore \quad n(M \cup H) = n(M) + n(H) - n(M \cap H)$

$= 400 + 200 - 50 = 550$

But, $M \cup H \subseteq U$. Therefore,

$n(M \cup H) \leq n(U)$

$\Rightarrow \quad n(M \cup H) \leq 500$

This is a contradiction. So, the given data is incorrect.

EXAMPLE 7 *If A and B be two sets containing 3 and 6 elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.*

SOLUTION We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

This shows that $n(A \cup B)$ is minimum or maximum according as $n(A \cap B)$ is maximum or minimum respectively.

CASE I *When $n(A \cap B)$ is minimum, i.e. $n(A \cap B) = 0$.*

This is possible only when $A \cap B = \phi$. In this case,

$n(A \cup B) = n(A) + n(B) - 0 = n(A) + n(B) = 3 + 6 = 9.$

So, maximum number of elements in $A \cup B$ is 9.

CASE II *When $n(A \cap B)$ is maximum.*

This is possible only when $A \subseteq B$. In this case, $n(A \cap B) = 3.$

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B) = (3 + 6 - 3) = 6.$

So, minimum number of elements in $A \cup B$ is 6.

EXAMPLE 8 *A market research group conducted a survey of 2000 consumers and reported that 1720 consumers liked product P_1 and 1450 consumers liked product P_2. What is the least number that must have liked both the products?*

SOLUTION Let U be the set of all consumers who were questioned, A be the set of consumers who liked product P_1 and B be the set of consumers who liked the product P_2.

It is given that

$n(U) = 2000, n(A) = 1720, n(B) = 1450.$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B) = 1720 + 1450 - n(A \cap B) = 3170 - n(A \cap B)$

Since, $\quad A \cup B \subset U$

$\therefore \quad n(A \cup B) \leq n(U)$

$\Rightarrow \quad 3170 - n(A \cap B) \leq 2000$

$\Rightarrow \quad 3170 - 2000 \leq n(A \cap B)$

$\Rightarrow \quad n(A \cap B) \geq 1170$

Thus, the least value of $n(A \cap B)$ is 1170.

Hence, the least number of consumer who liked both the products is 1170.

EXAMPLE 9 *There are 40 students in a chemistry class and 60 students in a physics class. Find the number of students which are either in physics class or chemistry class in the following cases:*

(i) the two classes meet at the same hour.

(ii) the two classes meet at different hours and 20 students are enrolled in both the subjects.

SOLUTION Let A be the set of students in chemistry class and B be the set of students in physics class.

It is given that

$n(A) = 40$ and $n(B) = 60.$

(i) If two classes meet at the same hour, then there will not be a common student sitting in both the classes. Therefore,

$n(A \cap B) = 0$

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad n(A \cup B) = 40 + 60 - 0 = 100$

(ii) If two classes meet at different timings then there can be some students attending both the classes. It is given that the number of such students is 20 i.e.

$n(A \cap B) = 20$

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 40 + 60 - 20 = 80.$

EXAMPLE 10 *If A, B and C are three sets and U is the universal set such that $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Find $n(A' \cap B')$.*

SOLUTION We have, $A' \cap B' = (A \cup B)'$

$\therefore$ $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 700 - (200 + 300 - 100) = 300.$

EXAMPLE 11 *In a survey of 700 students in a college, 180 were listed as drinking Limca, 275 as drinking Miranda and 95 were listed as both drinking Limca as well as Miranda. Find how many students were drinking neither Limca nor Miranda.*

SOLUTION Let U be the set of all surveyed students, A denote the set of students drinking Limca and B be the set students drinking Miranda.

It is given that

$n(U) = 700$, $n(A) = 180$, $n(B) = 275$ and $n(A \cap B) = 95$

We have to find.

$n(A' \cap B')$

Now, $n(A' \cap B') = n(A \cup B)'$

$= n(U) - n(A \cup B)$

$= n(U) - [n(A) + n(B) - n(A \cap B)]$

$= 700 - [180 + 275 - 95]$

$= 700 - 360$

$= 340$

EXAMPLE 12 *A survey shows that 63% of the Americans like cheese whereas 76% like apples. If x% of the Americans like both cheese and apples, find the value of x.*

SOLUTION Let A denote the set of Americans who like cheese and let B denote those who like apples. Let the population of America is 100. Then,

$n(A) = 63$, $n(B) = 76$.

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow$ $n(A \cup B) = 63 + 76 - n(A \cap B)$

$\Rightarrow$ $n(A \cup B) + n(A \cap B) = 139$

$\Rightarrow$ $n(A \cap B) = 139 - n(A \cup B)$.

But, $n(A \cup B) \le 100$, so that

$-n(A \cup B) \ge -100$

$\Rightarrow$ $139 - n(A \cup B) \ge 139 - 100$

$\Rightarrow$ $139 - n(A \cup B) \ge 39 \Rightarrow n(A \cap B) \ge 39$ ...(i)

Now, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\Rightarrow$ $n(A \cap B) \le n(A)$ and $n(A \cap B) \le n(B)$

$\Rightarrow$ $n(A \cap B) \le 63$ and $n(A \cap B) \le 76$

$\Rightarrow$ $n(A \cap B) \le 63$ ...(ii)

From (i) and (ii), we have 39

$39 \le n(A \cap B) \le 63 \Rightarrow 39 \le x \le 63.$

EXAMPLE 13 *In a class of 35 students, 17 have taken mathematics, 10 have taken mathematics but not economics. Find the number of students who have taken both mathematics and economics and the number of students who have taken economics but not mathematics, if it is given that each student has taken either mathematics or economics or both.*

SOLUTION Let A denote the set of students who have taken mathematics and B be the set of students who have taken economics.

It is given that

$n(A \cup B) = 35$, $n(A) = 17$ and $n(A - B) = 10$.

Now, $n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow$ $17 = 10 + n(A \cap B)$

$\Rightarrow$ $n(A \cap B) = 7$

Thus, 7 students have taken both mathematics and economics.

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow$ $35 = 17 + n(B) - 7$

$\Rightarrow$ $n(B) = 25$

Now, $n(B) = n(B - A) + n(A \cap B)$

$\Rightarrow$ $25 = n(B - A) + 7$

$\Rightarrow$ $n(B - A) = 18.$

Thus, 18 students have taken economics but not mathematics.

EXAMPLE 14 *In a town of 10,000 families it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C. 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three news papers, find the number of families which buy (i) A only (ii) B only (iii) none of A, B and C.*

SOLUTION Let P, Q and R be the sets of families buying newspaper A, B and C respectively. Let U be the universal set. Then,

$n(P) = 40\%$ of $10{,}000 = 4000$, $n(Q) = 20\%$ of $10{,}000 = 2000$,

$n(R) = 10\%$ of $10{,}000 = 1000$, $n(P \cap Q) = 5\%$ of $10{,}000 = 500$,

$n(Q \cap R) = 3\%$ of $10{,}000 = 300$, $n(R \cap P) = 4\%$ of $10{,}000 = 400$

$n(P \cap Q \cap R) = 2\%$ of $10{,}000 = 200$ and $n(U) = 10{,}000$.

(i) Required number $= n(P \cap Q' \cap R') = n(P \cap (Q \cup R)')$

$= n(P) - n[P \cap (Q \cup R)]$ $[\because n(A \cap B') = n(A) - n(A \cap B)]$

$= n(P) - n[(P \cap Q) \cup (P \cap R)]$

$= n(P) - [n(P \cap Q) + n(P \cap R) - n[(P \cap Q) \cap (P \cap R)]]$

$= n(P) - [n(P \cap Q) + n(P \cap R) - n(P \cap Q \cap R)]$

$= 4000 - [500 + 400 - 200] = 3300$

(ii) Required number $= n(P' \cap Q \cap R') = n(Q \cap P' \cap R')$

$= n(Q \cap (P \cup R)')$

$= n(Q) - n(Q \cap (P \cup R))$ $[\because n(A \cap B') = n(A) - n(A \cap B)]$

$= n(Q) - n[(Q \cap P) \cup (Q \cap R)]$

$= n(Q) - [n(Q \cap P) + n(Q \cap R) - n[(Q \cap P) \cap (Q \cap R)]]$

$= n(Q) - [n(P \cap Q) + n(Q \cap R) - n(P \cap Q \cap R)]$

$= 2000 - [500 + 300 - 200] = 1400$

(iii) Required number $= n(P' \cap Q' \cap R') = n[(P \cup Q \cup R)']$

$= n(U) - n(P \cup Q \cup R)$

$= n(U) - [n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(R \cap P) + n(P \cap Q \cap R)]$

$= 10000 - [4000 + 2000 + 1000 - 500 - 300 - 400 + 200] = 4000$

EXAMPLE 15 *A college awarded 38 medals in Football, 15 in Basketball and 20 to Cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports?*

SOLUTION Let F denote the set of men who received medals in Football, B the set of men who received medals in Basketball and C the set of men who received medals in Cricket. Then, we have

$$n(F)=38,\ n(B)=15,\ n(C)=20,\ n(F\cup B\cup C)=58 \text{ and } n(F\cap B\cap C)=3$$

Now,

$$n(F\cup B\cup C)=n(F)+n(B)+n(C)-n(F\cap B)-n(B\cap C)-n(F\cap C)+n(F\cap B\cap C)$$

$$\Rightarrow\ 58=38+15+20-n(F\cap B)-n(B\cap C)-n(F\cap C)+3$$

$$\Rightarrow\ n(F\cap B)+n(B\cap C)+n(F\cap C)=76-58=18$$

Now,

Number of men who received medals in exactly two of the three sports

$$=n(F\cap B)+n(B\cap C)+n(F\cap C)-3n(F\cap B\cap C)$$

$$=18-3\times 3$$

$$=9.$$

Thus, 9 men received medals in exactly two of the three sports.

EXAMPLE 36 *In a survey of 25 students, it was found that 15 had taken mathematics, 12 had taken physics and 11 had taken chemistry, 5 had taken mathematics and chemistry, 9 had taken mathematics and physics, 4 had taken physics and chemistry and 3 had taken all the three subjects. Find the number of students that had*

(i) *only chemistry.*
(ii) *only mathematics.*
(iii) *only physics.*
(iv) *physics and chemistry but not mathematics.*
(v) *mathematics and physics but not chemistry.*
(vi) *only one of the subjects.*
(vii) *at least one of the three subjects.*
(viii) *none of the subjects.*

SOLUTION Let M denote the set of students who had taken mathematics, P the set of students who had taken physics and C the set of students who had taken chemistry. Then, we have

$$n(U)=25,\ n(M)=15,\ n(P)=12,\ n(C)=11,\ n(M\cap C)=5,$$
$$n(M\cap P)=9,\ n(P\cap C)=4,\ n(M\cap P\cap C)=3$$

(i) Required number of students

$$=n(M'\cap P'\cap C)$$
$$=n((M\cup P)'\cap C)$$
$$=n(C)-n((M\cup P)\cap C) \qquad [\because n(A\cap B')=n(A)-n(A\cap B)]$$
$$=n(C)-n((M\cap C)\cup(P\cap C))$$
$$=n(C)-[n(M\cap C)+n(P\cap C)-n(M\cap P\cap C)]$$
$$=11-[5+4-3]=5$$

(ii) Required number of students

$$=n(M\cap P'\cap C')$$
$$=n(M\cap(P\cup C)')$$
$$=n(M)-n(M\cap(P\cup C))$$
$$=n(M)-n((M\cap P)\cup(M\cap C))$$
$$=n(M)-[n(M\cap P)+n(M\cap C)-n(M\cap P\cap C)]$$
$$=15-[9+5-3]=4$$

(iii) Required number of students

$$=n(P\cap M'\cap C')$$
$$=n(P\cap(M\cup C)')$$
$$=n(P)-n(P\cap(M\cup C))$$
$$=n(P)-n((P\cap M)\cup(P\cap C))$$
$$=n(P)-[n(P\cap M)+n(P\cap C)-n(P\cap M\cap C)]$$
$$=12-[9+4-3]=2$$

(iv) Required number of students

$$=n(P\cap C\cap M')$$
$$=n(P\cap C)-n(P\cap C\cap M) \qquad [\because n(A\cap B')=n(A)-n(A\cap B)]$$
$$=4-3=1$$

(v) Required number of students

$$=n(M\cap P\cap C')$$
$$=n(M\cap P)-n(M\cap P\cap C)=9-3=6$$

(vi) Required number of students

$$=n(M)+n(P)+n(C)-2[n(M\cap P)+n(P\cap C)+n(M\cap C)]+3n(M\cap P\cap C)$$
$$=15+12+11-2[9+4+5]+3\times 3$$
$$=38-36+9=11$$

(vii) Required number of students

$$=n(M\cup P\cup C)$$
$$=n(M)+n(P)+n(C)-n(M\cap P)-n(P\cap C)-n(M\cap C)+n(M\cap P\cap C)$$
$$=15+12+11-9-4-5+3=23$$

(viii) Required number of students

$$=n(M'\cap P'\cap C')$$
$$=n(M\cup P\cup C)'$$
$$=n(U)-n(M\cup P\cup C)=25-23=2$$

ALITER Consider the Venn diagram shown in Fig. 1.13. a, b, c, d, e, f, g denote the number of students in the respective regions.

From the Venn-daigram, we have

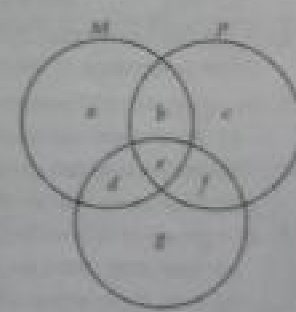


Fig. 1.13

$$n(M)=a+b+d+e,$$
$$n(P)=b+c+e+f,$$
$$n(C)=d+e+f+g,$$
$$n(M\cap P)=b+e$$
$$n(P\cap C)=e+f$$
$$n(M\cap C)=d+e$$

and, $\quad n(M\cap P\cap C)=e$

$$n(M\cap P\cap C)=3\Rightarrow e=3$$
$$n(M\cap P)=9\Rightarrow b+e=9\Rightarrow b+3=9\Rightarrow b=6$$
$$n(P\cap C)=4\Rightarrow e+f=4\Rightarrow 3+f=4\Rightarrow f=1$$
$$n(M\cap C)=5\Rightarrow d+e=5\Rightarrow d+3=5\Rightarrow d=2$$

$$n(M) = 15 \Rightarrow a+b+d+e = 15$$
$$\Rightarrow a+6+2+3 = 15$$
$$\Rightarrow a = 4$$
$$n(P) = 12 \Rightarrow b+c+e+f = 12$$
$$\Rightarrow 6+c+3+1 = 12$$
$$\Rightarrow c = 2$$
$$n(C) = 11 \Rightarrow d+e+f+g = 11$$
$$\Rightarrow 2+3+1+g = 11$$
$$\Rightarrow g = 5$$

Now,

(i) Required number of students $= g = 5$
(ii) Required number of students $= a = 4$
(iii) Required number of students $= c = 2$
(iv) Required number of students $= f = 1$
(v) Required number of students $= b = 6$
(vi) Required number of students $= a+c+g = 4+2+5 = 11$
(vii) Required number of students $= a+b+c+d+e+f+g = 23$
(viii) Required number of students $= 25-(a+b+c+d+e+f+g) = 25-23 = 2$

EXERCISE 1.8

1. If A and B are two sets such that $n(A \cup B) = 50$, $n(A) = 28$ and $n(B) = 32$, find $n(A \cap B)$.
2. If P and Q are two sets such that P has 40 elements, $P \cup Q$ has 60 elements and $P \cap Q$ has 10 elements, how many elements does Q have?
3. In a school there are 20 teachers who teach mathematics or physics. Of these, 12 teach mathematics and 4 teach physics and mathematics. How many teach physics?
4. In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many like both coffee and tea?
5. Let A and B be two sets such that: $n(A) = 20$, $n(A \cup B) = 42$ and $n(A \cap B) = 4$. Find
   (i) $n(B)$ (ii) $n(A-B)$ (iii) $n(B-A)$
6. A survey shows that 76% of the Indians like oranges, whereas 62% like bananas. What percentage of the Indians like both oranges and bananas?
7. In a group of 950 persons, 750 can speak Hindi and 460 can speak English. Find:
   (i) how many can speak both Hindi and English;
   (ii) how many can speak Hindi only;
   (iii) how many can speak English only.
8. In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find:
   (i) how may drink tea and coffee both;
   (ii) how many drink coffee but not tea.
9. In a survey of 60 people, it was found that 25 people read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find:
   (i) the numbers of people who read at least one of the newspapers.
   (ii) the number of people who read exactly one newspaper.

10. Of the members of three athletic teams in a certain school, 21 are in the basketball team, 26 in hockey team and 29 in the football team. 14 play hockey and basket ball, 15 play hockey and football, 12 play football and basketball and 8 play all the three games. How many members are there in all?
11. In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only? How many can speak Bengali? How many can speak both Hindi and Bengali?
12. A survey of 500 television viewers produced the following information; 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games? How many watch exactly one of the three games?
13. In a survey of 100 persons it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazines A and B, 10 read magazines A and C, 5 read magazines B and C and 3 read all the three magazines. Find:
    (i) How many read none of three magazines?
    (ii) How many read magazine C only?
14. In a survey of 100 students, the number of students studying the various languages were found to be: English only 18, English but not Hindi 23, English and Sanskrit 8, English 26, Sanskrit 48, Sanskrit and Hindi 8, no language 24. Find:
    (i) How many students were stydying Hindi?
    (ii) How many students were studying English and Hindi?
15. In a survey it was found that 21 persons liked product P_1, 26 liked product P_2 and 29 liked product P_3. If 14 persons liked products P_1 and P_2; 12 persons liked product P_3 and P_1; 14 persons liked products P_2 and P_3 and 8 liked all the three products. Find how many liked product P_3 only.

ANSWERS

1. 10 2. 30 3. 12 4. 19 5. (i) 26 (ii) 16 (iii) 22
6. 38% 7. (i) 260 (ii) 490 (iii) 200 8. (i) 16 (ii) 20
9. (i) 52 (ii) 30 10. 43 11. (i) 600 (ii) 250 (iii) 150
12. 20, 325 13. (i) 20 (ii) 30 14. (i) 18 (ii) 3 15. 11

HINTS TO SELECTED PROBLEMS

5. (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow n(B) = 26$
   (ii) $n(A-B) = n(A) - n(A \cap B) \Rightarrow n(A-B) = 16$
   (iii) $n(B-A) = n(B) - n(A \cap B) \Rightarrow 22$
7. (iii) Let A and B denote the sets of persons who can speak Hindi and English respectively. Then, $n(A \cup B) = 950$, $n(A) = 750$ and $n(B) = 460$.
   (i) $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 750 + 460 - 950 = 260$
   (ii) Required number $= n(A-B) = n(A) - n(A \cap B)$
   (iii) Required number $= n(B-A) = n(B) - n(A \cap B)$.
8. (ii) Let A and B be sets of persons who drink tea and coffee respectivley. Then $n(A \cup B) = 50$, $n(A-B) = 14$, $n(A) = 30$.
   (i) $n(A-B) = 14 \Rightarrow n(A) - n(A \cap B) = 14 \Rightarrow n(A \cap B) = n(A) - 14 = 30 - 14 = 16$
   (ii) Required number $= n(B-A) = n(B) - n(A \cap B)$
   Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
   $\Rightarrow 50 = 30 + n(B) - 16$

$\Rightarrow \quad n(B) = 36.$

So, $n(B-A) = n(B) - n(A \cap B) \Rightarrow n(B-A) = 36 - 16 = 20$

10. Let A, B and C be the sets of members of basketball, hockey and football teams respectively. Then $n(A) = 21$, $n(B) = 26$, $n(C) = 29$, $n(A \cap B) = 14$, $n(B \cap C) = 15$, $n(A \cap C) = 12$ and $n(A \cap B \cap C) = 8$.

    Required number $= n(A \cup B \cup C)$

    $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

11. Let A and B be the sets of persons who can speak Hindi and Bengali respectively. $n(A \cup B) = 1000$, $n(A) = 750$, $n(B) = 400$.

    No. of persons who can speak Hindi only $= n(A - B) = n(A) - n(A \cap B)$

    No. of persons who can speak Bengali only $= n(B - A) = n(B) - n(B \cap A)$

    No. of persons who can speak both Hindi and Bengali

    $$= n(A \cap B) = n(A) + n(B) - n(A \cup B).$$

12. N = Total number of television viewers $= 500$, $n(F) = 285$, $n(H) = 195$, $n(B) = 115$, $n(F \cap B) = 45$, $n(F \cap H) = 70$, $n(H \cap B) = 50$, $n(F' \cap H' \cap B') = 50$.

    Now,

    $n(F' \cap H' \cap B') = 50$

    $\Rightarrow n[(F \cup H \cup B)'] = 50$

    $\Rightarrow N - n(F \cup H \cup B) = 50$

    $\Rightarrow 500 - [n(F) + n(H) + n(B) - n(F \cap H) - n(F \cap B) - n(H \cap B) + n(F \cap H \cap B)] = 50$

    $\Rightarrow n(F \cap H \cap B) = 500 - 285 - 195 - 115 + 70 + 50 + 45 - 50 = 20.$

    $\therefore$ Required number $= n(F \cap H \cap B) = 20$

    Required number $= n(F \cap H' \cap B') + n(F' \cap H' \cap B) + n(F' \cap H \cap B')]$

    $= n(F) + n(H) + n(B) - 2[n(F \cap H) + n(H \cap B) + n(B \cap F)] + 3n(F \cap H \cap B)$

14. We have, $a = 18$, $a + b = 23$, $d + e = 8$, $a + b + d + e = 26$, $d + e + f + g = 48$, $e + f = 8$, $a + b + c + d + e + f + g = 100 - 24 = 76$

    $\therefore a = 18, b = 0, c = 10, d = 5, e = 3, f = 5$ and $g = 35$

    (i) $n(H) = b + c + e + f = 18$ (ii) $n(H \cap F) = b + e = 3$

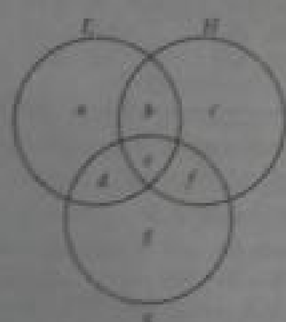


Fig. 1.14

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If a set contains n elements, then write the number of elements in its power set.
2. Write the number of elements in the power set of null set.

3. Let $A = \{x : x \in N, x$ is a multiple of $3\}$ and $B = \{x : x \in N$ and x is a multiple of $5\}$. Write $A \cap B$.
4. Let A and B be two sets having 3 and 6 elements respectively. Write the minimum number of elements that $A \cup B$ can have.
5. If $A = \{x \in C : x^2 = 1\}$ and $B = \{x \in C : x^4 = 1\}$, then write $A - B$ and $B - A$.
6. If A and B are two sets such that $A \subset B$, then write $B' - A'$ in terms of A and B.
7. Let A and B be two sets having 4 and 7 elements respectively. Then write the maximum number of elements that $A \cup B$ can have.
8. If $A = \{(x, y) : y = \frac{1}{x}, 0 \neq x \in R\}$ and $B = \{(x, y) : y = -x, x \in R\}$, then write $A \cap B$.
9. If $A = \{(x, y) : y = e^x, x \in R\}$ and $B = \{(x, y) : y = e^{-x}, x \in R\}$, then write $A \cap B$.
10. If A and B are two sets such that $n(A) = 20$, $n(B) = 25$ and $n(A \cup B) = 40$, then write $n(A \cap B)$.
11. If A and B are two sets such that $n(A) = 115$, $n(B) = 326$, $n(A - B) = 47$, then write $n(A \cup B)$.

ANSWERS

1. 2^n 2. 1 3. $\{x : x \in N, x$ is a multiple of $15\}$ 4. 6
5. $A - B = \phi, B - A = \{i, -i\}$ 6. ϕ 7. 11 8. ϕ 9. $\{(0, 1)\}$
10. 5 11. 373

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. For any set A, $(A')'$ is equal to
   (a) A' (b) A (c) ϕ (d) none of these.
2. Let A and B be two sets in the same universal set. Then, $A - B =$
   (a) $A \cap B$ (b) $A' \cap B$ (c) $A \cap B'$ (d) none of these.
3. The number of subsets of a set containing n elements is
   (a) n (b) $2^n - 1$ (c) n^2 (d) 2^n
4. For any two sets A and B, $A \cap (A \cup B) =$
   (a) A (b) B (c) ϕ (d) none of these.
5. If $A = \{1, 3, 5, B\}$ and $B = \{2, 4\}$, then
   (a) $4 \in A$ (b) $\{4\} \subset A$ (c) $B \subset A$ (d) none of these.
6. The symmetric difference of A and B is
   (a) $(A - B) \cap (B - A)$ (b) $(A - B) \cup (B - A)$
   (c) $(A \cup B) - (A \cap B)$ (d) $\{(A \cup B) - A\} \cup \{(A \cup B) - B\}$
7. The symmetric difference of $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ is
   (a) $\{1, 2\}$ (b) $\{1, 2, 4, 5\}$ (c) $\{4, 3\}$ (d) $\{2, 5, 1, 4, 3\}$
8. For any two sets A and B, $(A - B) \cup (B - A) =$
   (a) $(A - B) \cup A$ (b) $(B - A) \cup B$
   (c) $(A \cup B) - (A \cap B)$ (d) $(A \cup B) \cap (A \cap B)$.
9. Which of the following statement is false :
   (a) $A - B = A \cap B'$ (b) $A - B = A - (A \cap B)$

(c) $A - B = A - B'$ (d) $A - B = (A \cup B) - B$.

10. For any three sets A, B and C

(a) $A \cap (B - C) = (A \cap B) - (A \cap C)$ (b) $A \cap (B - C) = (A \cap B) - C$

(c) $A \cup (B - C) = (A \cup B) \cap (A \cup C')$ (d) $A \cup (B - C) = (A \cup B) - (A \cup C)$.

11. Let $A = \{x : x \in R, x \geq 4\}$ and $B = \{x \in R : x < 5\}$. Then, $A \cap B =$

(a) $(4, 5]$ (b) $(4, 5)$ (c) $[4, 5)$ (d) $[4, 5]$

12. Let U be the universal set containing 700 elements. If A, B are sub-sets of U such that $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Then, $n(A' \cap B') =$

(a) 400 (b) 600 (c) 300 (d) none of these.

13. Let A and B be two sets such that $n(A) = 16$, $n(B) = 14$, $n(A \cup B) = 25$. Then, $n(A \cap B)$ is equal to

(a) 30 (b) 50 (c) 5 (d) none of these

14. If $A = \{1, 2, 3, 4, 5\}$, then the number of proper subsets of A is

(a) 120 (b) 30 (c) 31 (d) 32

15. In set-builder method the null set is represented by

(a) $\{ \}$ (b) Φ (c) $\{x : x \neq x\}$ (d) $\{x : x = x\}$

16. If A and B are two disjoint sets, then $n(A \cup B)$ is equal to

(a) $n(A) + n(B)$ (b) $n(A) + n(B) - n(A \cap B)$

(c) $n(A) + n(B) + n(A \cap B)$ (d) $n(A)\, n(B)$ (e) $n(A) - n(B)$

17. For two sets $A \cup B = A$ iff

(a) $B \subseteq A$ (b) $A \subseteq B$ (c) $A \neq B$ (d) $A = B$

18. If A and B are two sets such that $n(A) = 70$, $n(B) = 60$, $n(A \cup B) = 110$, then $n(A \cap B)$ is equal to

(a) 240 (b) 50 (c) 40 (d) 20

19. If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to

(a) A (b) B (c) Φ (d) $A \cap B^c$

20. If $A = \{x : x$ is a multiple of 3$\}$ and, $B = \{x : x$ is a multiple of 5$\}$, then $A - B$ is

(a) $A \cap B$ (b) $A \cap \overline{B}$ (c) $\overline{A} \cap \overline{B}$ (d) $\overline{A \cap B}$

21. In a city 20% of the population travels by car, 50% travels by bus and 10% travels by both car and bus. Then, persons travelling by car or bus is

(a) 80% (b) 40% (c) 60% (d) 70%

22. If $A \cap B = B$, then

(a) $A \subset B$ (b) $B \subset A$ (c) $A = \Phi$ (d) $B = \Phi$

23. An investigator interviewed 100 students to determine the performance of three drinks: milk, coffee and tea. The investigator reported that 10 students take all three drinks milk, coffee and tea; 20 students take milk and coffee; 25 students take milk and tea; 12 students take milk only; 5 students take coffee only and 8 students take tea only. Then the number of students who did not take any of three drinks is

(a) 10 (b) 20 (c) 25 (d) 30

24. Two finite sets have m and n elements. The number of elements in the power set of first set is 48 more than the total number of elements in power set of the second set.

Then, the values of m and n are :

(a) 7, 6 (b) 6, 3 (c) 6, 4 (d) 7, 4 (e) 3, 7

25. In a class of 175 students the following data shows the number of students opting one or more subjects. Mathematics 100; Physics 70; Chemistry 40; Mathematics and Physics 30; Mathematics and Chemistry 28; Physics and Chemistry 23; Mathematics, Physics and Chemistry 18. How many students have offered Mathematics alone ?

(a) 35 (b) 48 (c) 60 (d) 22 (e) 30

ANSWERS

1. (b)	2. (c)	3. (d)	4. (a)	5. (d)	6. (b)	7. (b)	8. (c)
9. (c)	10. (a), (b), (c)		11. (c)	12. (c)	13. (c)	14. (c)	15. (c)
16. (a)	17. (a)	18. (d)	19. (d)	20. (b)	21. (c)	22. (b)	23. (b)
24. (c)	25. (c)						

SUMMARY

1. A set is a well defined collection of objects.
2. A set is described either in set builder form or tabular form.
3. A set consisting of no element is called the null set and is denoted by ϕ.
4. A set consisting of a single element is called a singleton set.
5. A set consisting of a definite number of elements is called a finite set, otherwise the set is called an infinite set.
6. The number of elements in a finite set A is called its cardinal number or order and is denoted by $n(A)$.
7. Two sets A and B are equal if they have exactly the same elements.
8. A set A is said to be a subset of a set B, if every element of A is also an element of B.
9. If a, b are real numbers such that $a < b$, then the set
   (i) $\{x : x \in R$ and $a \leq x \leq b\}$ is called the closed interval $[a, b]$
   (ii) $\{x : x \in R$ and $a < x < b\}$ is called the open interval (a, b)
   (iii) $\{x : x \in R$ and $a \leq x < b\}$ is called the semi-open or semi-closed interval $[a, b)$.
   (iv) $\{x : x \in R$ and $a < x \leq b\}$ is called the semi-open or semi-closed interval $(a, b]$.
10. The total number of subsets of a finite set consisting of n elements is 2^n.
11. The collection of all subsets of a set A is called the power set of A and is denoted by $P(A)$.
12. The union of two sets A and B is the set of all those elements which are either in A or in B or in both and is denoted by $A \cup B$.
    Thus, $A \cup B = \{x : x \in A$ or $x \in B\}$
13. The intersection of two sets A and B is the set of all those elements which are common to both A and B and is denoted by $A \cap B$.
    Thus, $A \cap B = \{x : x \in A$ and $x \in B\}$.
14. The difference $A - B$ of two sets A and B is the set of all those elements of A which do not belong to B i.e.

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Similarly,

$$B - A = \{x : x \in B \text{ and } x \notin A\}.$$

15. The symmetric difference of two sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.
16. The complement of a subset A of universal set U is the set of all those elements of U which are not the elements of A.

    The complement of A is denoted by A' or A^c.
17. For any three sets A, B and C, we have
    (i) $A \cup A = A$ and $A \cap A = A$ (Idempotent laws)
    (ii) $A \cup \phi = A$ and $A \cap U = A$ (Identity laws)
    (iii) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (Commutative laws)
    (iv) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative laws)
    (v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive laws)
    (vi) $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ (De' Morgan's laws)
18. If A, B and C are finite sets and U be the finite universal set, then
    (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
    (ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets
    (iii) $n(A - B) = n(A) - n(A \cap B)$ i.e., $n(A - B) + n(A \cap B) = n(A)$
    (iv) $n(A \Delta B) = n(A - B) + n(B - A) = n(A) + n(B) - 2n(A \cap B)$
    (v) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$
    (vi) Number of elements in exactly two of sets A, B and C $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
    (vii) Number of elements in exactly one of sets A, B and C $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

2

RELATIONS

2.1 INTRODUCTION

In previous chapter, we have discussed various operations on sets to create more sets out of given sets. In this chapter, we shall study one more operation which is known as the cartesian product of sets. This will finally enable us to introduce the concept of relation.

2.2 ORDERED PAIRS

ORDERED PAIR *An ordered pair consists of two objects or elements in a given fixed order.*

For example, if A and B are any two sets, then by an ordered pair of elements we mean a pair (a, b) in that order, where $a \in A$, $b \in B$.

NOTE *An ordered pair is not a set consisting of two elements. The ordering of the two elements in an ordered pair is important and the two elements need not be distinct.*

ILLUSTRATION 1 *The position of a point in a two dimensional plane in cartesian coordinates is represented by an ordered pair. Accordingly, the ordered pairs (1, 3), (2, 4), (2, 3) and (3, 2) represents different points in a plane.*

EQUALITY OF ORDERED PAIRS: *Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff*

$$a_1 = a_2 \text{ and } b_1 = b_2$$

i.e., $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

It is evident from this definition that $(1, 2) \neq (2, 1)$ and $(1, 1) \neq (2, 2)$.

ILLUSTRATION 2 *Find the values of a and b, if $(3a - 2, b + 3) = (2a - 1, 3)$.*

SOLUTION By the definition of equality of ordered pairs, we have

$$(3a - 2, b + 3) = (2a - 1, 3)$$
$$\Leftrightarrow \quad 3a - 2 = 2a - 1 \text{ and } b + 3 = 3$$
$$\Leftrightarrow \quad a = 1 \text{ and } b = 0$$

2.3 CARTESIAN PRODUCT OF SETS

CARTESIAN PRODUCT OF SETS *Let A and B be any two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of the sets A and B and is denoted by $A \times B$.*

Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

If $A = \phi$ or $B = \phi$, then we define $A \times B = \phi$

ILLUSTRATION 1 *If $A = \{2, 4, 6\}$ and $B = \{1, 2\}$, then*

$$A \times B = \{2, 4, 6\} \times \{1, 2\} = \{(2, 1), (2, 2), (4, 1), (4, 2), (6, 1), (6, 2)\}$$

and $B \times A = \{1, 2\} \times \{2, 4, 6\} = \{(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6)\}$

It is evident from the above illustration that to write $A \times B$, we take an element from set A and form all ordered pairs with this element as first element and elements of B as second elements. Next we choose another element from A and corresponding to each element in B we form ordered pairs with this element as first element and elements of B as second elements. This process is continued till all elements of A are exhausted.

ILLUSTRATION 2 *If* $A = \{a, b\}$ *and* $B = \{1, 2, 3\}$, *find* $A \times B$, $B \times A$, $A \times A$, $B \times B$, *and* $(A \times B) \cap (B \times A)$.

SOLUTION We have,

$$A = \{a, b\} \text{ and } B = \{1, 2, 3\}$$

So,

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
$$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
$$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$
$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Clearly, $(A \times B) \cap (B \times A) = \phi$.

CARTESIAN PRODUCT OF THREE SETS *Let A, B and C be three sets. Then, $A \times B \times C$ is the set of all ordered triplets having first element from A, second element from B and third element from C.*

i.e., $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$

ILLUSTRATION 3 *If* $A = \{1, 2\}$, $B = \{3, 4\}$ *and* $C = \{4, 5, 6\}$. *Then,*

$$A \times B = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$
$$A \times B \times C = (A \times B) \times C$$
$$= \{(1, 3), (1, 4), (2, 3), (2, 4)\} \times \{4, 5, 6\}$$
$$= \{(1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 4), (1, 4, 5), (1, 4, 6), (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (2, 4, 5), (2, 4, 6)\}$$

NOTE *It should be noted that* $A \times B \times C = (A \times B) \times C = A \times (B \times C)$

If $A_1, A_2, A_3, \ldots, A_n$ are n sets, then the cartesian product $A_1 \times A_2 \times \ldots \times A_n$ of these n sets is the set of all n-tuples of the form $(a_1, a_2, a_3, \ldots, a_n)$, where $a_1 \in A, a_2 \in A_2, \ldots, a_n \in A_n$

i.e., $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, a_3, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots, a_n \in A_n\}$

2.3.1 NUMBER OF ELEMENTS IN THE CARTESIAN PRODUCT OF TWO SETS

THEOREM *If A and B are two finite sets, then* $n(A \times B) = n(A) \times n(B)$.

PROOF Let $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$ be two sets having m and n elements respectively. Then,

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots, (a_1, b_n)$$
$$(a_2, b_1), (a_2, b_2), (a_2, b_3), \ldots, (a_2, b_n)$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$(a_m, b_1), (a_m, b_2), (a_m, b_3), \ldots, (a_m, b_n)\}$$

Clearly, in the tabular representation of $A \times B$ there are m rows of ordered pairs and each row has n distinct ordered pairs.

So, $A \times B$ has mn elements.

Hence, $n(A \times B) = mn = n(A) \times n(B)$

REMARK (i) *If either A or B is an infinite set, then $A \times B$ is an infinite set.*

(ii) *If A, B, C are finite sets, then* $n(A \times B \times C) = n(A) \times n(B) \times n(C)$

2.3.2 GRAPHICAL REPRESENTATION OF CARTESIAN PRODUCT OF SETS

Let A and B be any two non-empty sets. To represent $A \times B$ graphically, we draw two mutually perpendicular lines, one horizontal and other vertical. On the horizontal line, we represent the elements of set A and on the vertical line, the elements of B. If $a \in A$, $b \in B$, we draw a vertical line through a and a horizontal line through b. These two lines will meet in a point which will denote the ordered pair (a, b). In this manner we mark points corresponding to each ordered pair in $A \times B$. The set of points so obtained represents $A \times B$ graphically as illustrated in the following example.

ILLUSTRATION *If* $A = \{1, 2, 3\}$ *and* $B = \{2, 4\}$, *find* $A \times B$ *and show it graphically.*

SOLUTION Clearly, $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$.

To represent $A \times B$ graphically, we draw two perpendicular lines OX and OY as shown in Fig. 2.1. Now we represent the set A by three points on OX and the set B by two points on OY. The set $A \times B$ is represented by the six points as shown in Fig. 2.1.

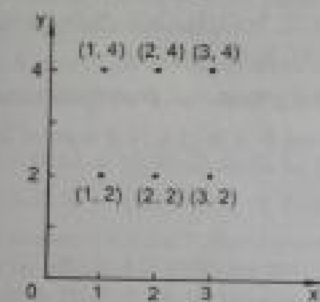


Fig. 2.1

2.3.3 DIAGRAMATIC REPRESENTATION OF CARTESIAN PRODUCT OF TWO SETS

In order to represent $A \times B$ by an arrow diagram, we first draw Venn diagrams representing sets A and B one opposite to the other as shown in Fig. 2.2. Now, we draw line segments starting from each element of A and terminating to each element of set B. If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then following figure gives the arrow diagram of $A \times B$.

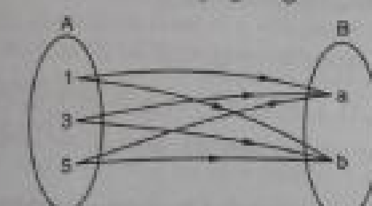


Fig. 2.2

ILLUSTRATIVE EXAMPLES

Type I ON EQUALITY OF ORDERED PAIRS

EXAMPLE 1 *Find x and y, if* $(x + 3, 5) = (6, 2x + y)$.

SOLUTION By the definition of equality of ordered pairs

$$(x + 3, 5) = (6, 2x + y)$$
$$\Rightarrow \quad x + 3 = 6 \text{ and } 5 = 2x + y$$
$$\Rightarrow \quad x = 3 \text{ and } 5 = 2x + y$$
$$\Rightarrow \quad x = 3, 5 = 6 + y$$
$$\Rightarrow \quad x = 3 \text{ and } y = -1$$

Type II **ON FINDING THE CARTESIAN PRODUCT OF TWO SETS**

EXAMPLE 2 *If $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$, find $A \times B$ and $B \times A$.*

SOLUTION We have,

$$A \times B = \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (6, 2), (6, 4)\}$$

and, $$B \times A = \{(2, 1), (2, 3), (2, 5), (2, 6), (4, 1), (4, 3), (4, 5), (4, 6)\}$$

EXAMPLE 3 *If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{1, 3, 5\}$, find*
(i) $A \times (B \cup C)$ (ii) $A \times (B \cap C)$ (iii) $(A \times B) \cap (A \times C)$

SOLUTION We have,

(i) $B \cup C = \{1, 3, 4, 5\}$

$\therefore$ $A \times (B \cup C) = \{1, 2, 3\} \times \{1, 3, 4, 5\}$

$= \{(1, 1), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 3), (3, 4), (3, 5)\}$

(ii) $B \cap C = \{3\}$. $A \times (B \cap C) = \{1, 2, 3\} \times \{3\} = \{(1, 3), (2, 3), (3, 3)\}$

(iii) $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$,

and, $A \times C = \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$

$\therefore$ $(A \times B) \cap (A \times C) = \{(1, 3), (2, 3), (3, 3)\}$

Type III **ON FINDING SETS A AND B WHEN $A \times B$ OR SOME ELEMENTS OF $A \times B$ ARE GIVEN**

EXAMPLE 4 *Let $A = \{1, 2, 3\}$ and $B = \{x : x \in N, x$ is prime less than $5\}$. Find $A \times B$ and $B \times A$.*

SOLUTION We have,

$$A = \{1, 2, 3\}$$

and, $B = \{x : x \in N, x$ is prime less than $5\} = \{2, 3\}$

$\therefore$ $A \times B = \{1, 2, 3\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$

and, $B \times A = \{2, 3\} \times \{1, 2, 3\} = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

EXAMPLE 5 *If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 5), (b, 1)\}$, find $B \times A$.*

SOLUTION Clearly, $B \times A$ can be obtained from $A \times B$ by interchanging the entries (or components) or ordered pair in $A \times B$.

$\therefore$ $B \times A = \{(1, a), (5, a), (2, a), (2, b), (5, b), (1, b)\}$

EXAMPLE 6 *If $A = \{1, 2\}$, form the set $A \times A \times A$.*

SOLUTION We have,

$$A \times A = \{(1,1), (1, 2), (2, 1), (2, 2)\}$$

$\therefore$ $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

EXAMPLE 7 *If R is the set of all real numbers, what do the cartesian products $R \times R$ and $R \times R \times R$ represent?*

SOLUTION The cartesian product of the set R of all real numbers with itself i.e. $R \times R$ is the set of all ordered pairs (x, y) where $x, y \in R$.

In other words,

$$R \times R = \{(x, y) : x, y \in R\}$$

Clearly, $R \times R$ is the set of all points in XY-plane.

The set $R \times R$ is also denoted by R^2.

Similarly, we have

$$R \times R \times R = \{(x, y, z) : x, y, z \in R\}$$

Clearly, it represents the set of all points in space.

The set $R \times R \times R$ is also denoted by R^3.

EXAMPLE 8 *Express $A = \{(a, b) : 2a + b = 5, a, b \in W\}$ as the set of ordered pairs.*

SOLUTION Here, W denotes the set of whole numbers (non-negative integers).

We have,

$$2a + b = 5$$

$\therefore$ $a = 0 \Rightarrow b = 5, a = 1 \Rightarrow b = 3, a = 2 \Rightarrow b = 1$

For $a \geq 3$, the values of b given by the above relation are not whole numbers.

$\therefore$ $A = \{(0, 5), (1, 3), (2, 1)\}$

EXAMPLE 9 *If $A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1), (a, 2), (b, 2)\}$, find A and B.*

SOLUTION Clearly, A is the set of all first entries in ordered pairs in $A \times B$ and B is the set of all second entries in ordered pairs in $A \times B$.

$\therefore$ $A = \{a, b\}$ and $B = \{1, 2, 3\}$

EXAMPLE 10 *Let A and B be two sets such that $A \times B$ consists of 6 elements. If three elements of $A \times B$ are : $(1, 4), (2, 6), (3, 6)$. Find $A \times B$ and $B \times A$.*

SOLUTION Since $(1, 4)$, $(2, 6)$ and $(3, 6)$ are elements of $A \times B$, it follows that 1, 2, 3 are elements of A and 4, 6 are elements of B. It is given that $A \times B$ has 6 elements. So, $A = \{1, 2, 3\}$ and $B = \{4, 6\}$.

Hence, $A \times B = \{1, 2, 3\} \times \{4, 6\} = \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$

and, $B \times A = \{4, 6\} \times \{1, 2, 3\} = \{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3)\}$

EXAMPLE 11 *The cartesian product $A \times A$ has 9 elements among which are found $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining elements of $A \times A$.*

SOLUTION Since $(-1, 0) \in A \times A$ and $(0, 1) \in A \times A$. Therefore,

$$(-1, 0) \in A \times A \Rightarrow -1, 0 \in A$$

and $$(0, 1) \in A \times A \Rightarrow 0, 1 \in A$$

$\therefore$ $-1, 0, 1 \in A$

It is given that A has exactly three elements. Hence, $A = \{-1, 0, 1\}$.

EXAMPLE 12 *Let A and B be two sets such that $n(A) = 5$ and $n(B) = 2$. If a, b, c, d, e are distinct and $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are in $A \times B$, find A and B.*

SOLUTION Since $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are elements of $A \times B$. Therefore,

$$a, b, c, d, e \in A \text{ and } 2, 3 \in B.$$

It is given that $n(A) = 5$ and $n(B) = 2$

$\therefore$ $a, b, c, d, e \in A$ and $n(A) = 5 \Rightarrow A = \{a, b, c, d, e\}$

$2, 3 \in B$ and $n(B) = 2 \Rightarrow B = \{2, 3\}$

Type IV **ON GRAPHICAL AND DIAGRAMATIC REPRESENTATION OF $A \times B$**

EXAMPLE 13 *Let $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$. Represent the following products graphically i.e. by lattices:*

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$

SOLUTION (i) We have,

$$A \times B = \{(-1, 2), (-1, 3), (3, 2), (3, 3), (4, 2), (4, 3)\}$$

In order to represent $A \times B$ graphically, we follow the following steps:

(a) Draw to mutually perpendicular line one horizontal and other vertical.

(b) On the horizontal line represent the elements of set A and on the vertical line represent the elements of set B.

(c) Draw vertical dotted lines through points representing elements of A on horizontal line and horizontal lines through points representing elements of B on the vertical line. Points of intersection of these lines will represent $A \times B$ graphically.

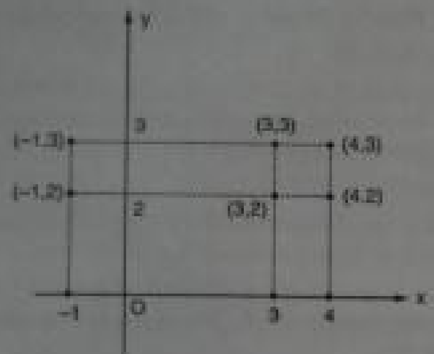


Fig. 2.3

(ii) We have,

$$B \times A = \{2, 3\} \times \{-1, 3, 4\} = \{(2, -1), (2, 3), (2, 4), (3, -1), (3, 3), (3, 4)\}$$

Here, we represent B on the horizontal line and A on vertical line.

Graphical representation of $B \times A$ is as shown below:

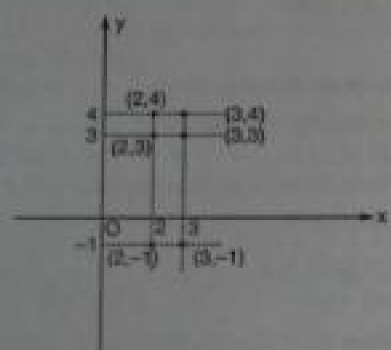


Fig. 2.4

(iii) We have,

$$A \times A = \{-1, 3, 4\} \times \{-1, 3, 4\}$$

$$= \{(-1, -1), (-1, 3), (-1, 4), (3, -1), (3, 3), (3, 4), (4, -1), (4, 3), (4, 4)\}$$

Graphical representation of $A \times A$ is shown in Fig. 2.5

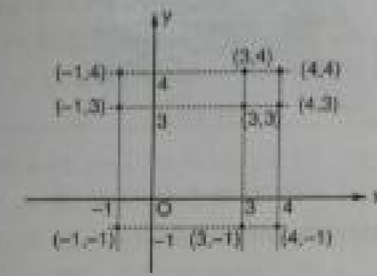


Fig. 2.5

EXAMPLE 14 *If $A = \{1, 3, 5\}$, $B = \{x, y\}$ represent the following products by arrow diagrams:*

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

SOLUTION (i) We have,

$$A \times B = \{1, 3, 5\} \times \{x, y\} = \{(1, x), (1, y), (3, x), (3, y), (5, x), (5, y)\}$$

Following arrow diagram represents $A \times B$.

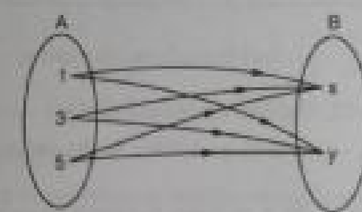


Fig. 2.6

(ii) We have,

$$B \times A = \{x, y\} \times \{1, 3, 5\} = \{(x, 1), (x, 3), (x, 5), (y, 1), (y, 3), (y, 5)\}$$

$B \times A$ has been represented by the following arrow diagram:

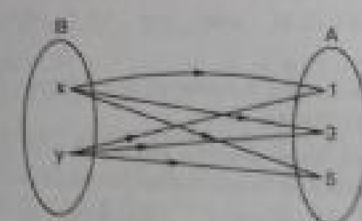


Fig. 2.7

(iii) We have,

$$A \times A = \{1, 3, 5\} \times \{1, 3, 5\} = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$$

It has been represented by the following arrow diagram:

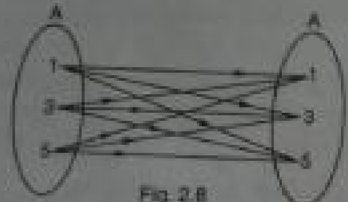


Fig. 2.8

(iv) We have,

$B \times B = \{x, y\} \times \{x, y\} = \{(x, x), (x, y), (y, x), (y, y)\}$

Following is the arrow diagram representing $B \times B$:

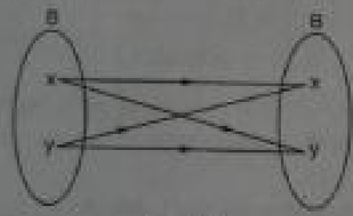


Fig. 2.9

EXERCISE 2.1

1. (i) If $\left(\frac{a}{3}+1, b-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of a and b.

   (ii) If $(x+1, 1)=(3, y-2)$, find the values of x and y.

2. If the ordered pairs $(x, -1)$ and $(5, y)$ belong to the set $\{(a, b) : b = 2a - 3\}$, find the values of x and y.

3. If $a \in \{-1, 2, 3, 4, 5\}$ and $b \in \{0, 3, 6\}$, write the set of all ordered pairs (a, b) such that $a+b=5$.

4. If $a \in \{2, 4, 6, 9\}$ and $b \in \{4, 6, 18, 27\}$, then form the set of all ordered pairs (a, b) such that a divides b and $a < b$.

5. If $A = \{1, 2\}$ and $B = \{1, 3\}$, find $A \times B$ and $B \times A$.

6. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Find $A \times B$ and show it graphically.

7. If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, what are $A \times B$, $B \times A$, $A \times A$, $B \times B$, and $(A \times B) \cap (B \times A)$?

8. If A and B are two sets having 3 elements in common. If $n(A) = 5$, $n(B) = 4$, find $n(A \times B)$ and $n[(A \times B) \cap (B \times A)]$.

9. Let A and B be two sets. Show that the sets $A \times B$ and $B \times A$ have an element in common *iff* the sets A and B have an element in common.

10. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$.
    If $(x, 1)$, $(y, 2)$, $(z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.

11. Let $A = \{1, 2, 3, 4\}$ and $R = \{(a, b) : a \in A, b \in A, a \text{ divides } b\}$. Write R explicitly.

12. If $A = \{-1, 1\}$, find $A \times A \times A$.

13. State whether each of the following statements are true or false. If the statement is false, re-write the given statement correctly:

    (i) If $P = \{m, n\}$ and $Q = \{n, m\}$, then $P \times Q = \{(m, n), (n, m)\}$

   (ii) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in B$ and $y \in A$.

   (iii) If $A = \{1, 2\}$, $B = \{3, 4\}$, then $A \times (B \cap \phi) = \phi$.

14. If $A = \{1, 2\}$, form the set $A \times A \times A$.

15. If $A = \{1, 2, 4\}$ and $B = \{1, 2, 3\}$, represent following sets graphically:
    (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

ANSWERS

1. (i) $a = 2, b = 1$ (ii) $x = 2, y = 3$ 2. $x = 1, y = 7$ 3. $\{(-1, 6), (2, 3), (5, 0)\}$
4. $\{(2, 4), (2, 6), (2, 18), (6, 18), (9, 18), (9, 27)\}$
5. $A \times B = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$ and $B \times A = \{(1, 1), (1, 2), (3, 1), (3, 2)\}$
6. $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$.
7. $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$
   $B \times A = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$
   $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   $B \times B = \{(2, 2), (2, 4), (4, 2), (4, 4)\}$
   $(A \times B) \cap (B \times A) = \{(2, 2)\}$
8. $n(A \times B) = 20$, $n[(A \times B) \cap (B \times A)] = 9$. 10. $A = \{x, y, z\}$, $B = \{1, 2\}$
11. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
12. $A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$
13. (i) F (ii) F (iii) T
14. $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

2.4 SOME USEFUL RESULTS

In this section, we intend to study some results on cartesian product of sets which are given as theorems.

THEOREM 1 *For any three sets A, B, C, prove that:*

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

PROOF (i) Let (a, b) be an arbitrary element of $A \times (B \cup C)$. Then,

$(a, b) \in A \times (B \cup C) \Rightarrow a \in A$ and $b \in B \cup C$ [by def.]

$\Rightarrow$ $a \in A$ and $(b \in B$ or $b \in C)$ [by def. of union]

$\Rightarrow$ $(a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$

$\Rightarrow$ $(a, b) \in A \times B$ or $(a, b) \in A \times C \Rightarrow (a, b) \in (A \times B) \cup (A \times C)$

$\therefore$ $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ ...(i)

Again, let (x, y) be an arbitrary element of $(A \times B) \cup (A \times C)$. Then,

$(x, y) \in (A \times B) \cup (A \times C) \Rightarrow (x, y) \in A \times B$ or $(x, y) \in A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C) \Rightarrow x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow$ $x \in A$ and $y \in (B \cup C) \Rightarrow (x, y) \in A \times (B \cup C)$

$\therefore$ $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ ...(ii)

Hence, from (i) and (ii), we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) Let (a, b) be an arbitrary element of $A \times (B \cap C)$. Then,

$(a, b) \in A \times (B \cap C) \Rightarrow a \in A$ and $b \in (B \cap C)$ [by def.]

$\Rightarrow$ $a \in A$ and $(b \in B$ and $b \in C) \Rightarrow (a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$

$\Rightarrow$ $(a, b) \in A \times B$ and $(a, b) \in A \times C$ [by def.]

$\Rightarrow$ $(a, b) \in (A \times B) \cap (A \times C)$

$\therefore$ $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ...(i)

Again, let (x, y) be an arbitrary element of $(A \times B) \cap (A \times C)$. Then,

$(x, y) \in (A \times B) \cap (A \times C) \Rightarrow (x, y) \in (A \times B)$ and $(x, y) \in A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C) \Rightarrow x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow$ $x \in A$ and $y \in (B \cap C) \Rightarrow (x, y) \in A \times (B \cap C)$

$\therefore$ $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ...(ii)

Hence, from (i) and (ii), we get

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ Q.E.D.

THEOREM 2 *For any three sets A, B, C prove that:*

$A \times (B - C) = (A \times B) - (A \times C)$

PROOF Let (a, b) be an arbitrary element of $A \times (B - C)$. Then,

$(a, b) \in A \times (B - C) \Rightarrow a \in A$ and $b \in (B - C) \Rightarrow a \in A$ and $(b \in B$ and $b \notin C)$

$\Rightarrow$ $(a \in A$ and $b \in B)$ and $(a \in A$ and $b \notin C)$

$\Rightarrow$ $(a, b) \in (A \times B)$ and $(a, b) \notin (A \times C) \Rightarrow (a, b) \in (A \times B) - (A \times C)$

$\therefore$ $A \times (B - C) \subseteq (A \times B) - (A \times C)$ ...(i)

Again, let (x, y) be an arbitrary element of $(A \times B) - (A \times C)$. Then,

$(x, y) \in (A \times B) - (A \times C) \Rightarrow (x, y) \in A \times B$ and $(x, y) \notin A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \notin C) \Rightarrow x \in A$ and $(y \in B$ and $y \notin C)$

$\Rightarrow$ $x \in A$ and $y \in (B - C) \Rightarrow (x, y) \in A \times (B - C)$

$\therefore$ $(A \times B) - (A \times C) \subseteq A \times (B - C)$ ...(ii)

Hence, from (i) and (ii), we get

$A \times (B - C) = (A \times B) - (A \times C)$ Q.E.D.

THEOREM 3 *If A and B are any two non-empty sets, then prove that:*

$A \times B = B \times A \Leftrightarrow A = B$

PROOF First, let $A = B$. Then we have to prove that $A \times B = B \times A$.

Now, $A = B \Rightarrow A \times B = A \times A$ and $B \times A = A \times A$ [$\because B = A$]

$\Rightarrow$ $A \times B = B \times A$

Conversely, let $A \times B = B \times A$. Then we have to prove that $A = B$.

Let x be an arbitrary element of A. Then,

$x \in A \Rightarrow (x, b) \in A \times B$ for all $b \in B$

$\Rightarrow (x, b) \in B \times A$ [$\because A \times B = B \times A$]

$\Rightarrow x \in B$ [By def.]

$\therefore$ $A \subseteq B$

Again, let y be an arbitrary element of B. Then,

$y \in B \Rightarrow (a, y) \in A \times B$ for all $a \in A$

$\Rightarrow (a, y) \in B \times A$ [$\because A \times B = B \times A$]

$\Rightarrow y \in A$ [By def.]

$\therefore$ $B \subseteq A$ Q.E.D.

Hence, $A = B$.

THEOREM 4 *If $A \subseteq B$, show that $A \times A \subseteq (A \times B) \cap (B \times A)$.*

PROOF Let (a, b) be an arbitrary element of $A \times A$. Then,

$(a, b) \in A \times A \Rightarrow a \in A$ and $b \in A \Rightarrow (a \in A, b \in A)$ and $(a \in A, b \in A)$

$\Rightarrow$ $(a \in A, b \in B)$ and $(a \in B, b \in A)$ [$\because A \subseteq B \therefore a, b \in A \Rightarrow a, b \in B$]

$\Rightarrow$ $(a, b) \in (A \times B)$ and $(a, b) \in (B \times A)$

$\Rightarrow$ $(a, b) \in (A \times B) \cap (B \times A)$

$\therefore$ $A \times A \subseteq (A \times B) \subseteq (B \times A)$

Hence, $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$. Q.E.D.

THEOREM 5 *If $A \subseteq B$, prove that $A \times C \subseteq B \times C$ for any set C.*

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,

$(a, b) \in A \times C \Rightarrow a \in A$ and $b \in C$

$\Rightarrow a \in B$ and $b \in C$ [$\because A \subseteq B \therefore a \in A \Rightarrow a \in B$]

$\Rightarrow (a, b) \in B \times C$

$\therefore$ $A \times C \subseteq B \times C$ Q.E.D.

THEOREM 6 *If $A \subseteq B$ and $C \subseteq D$, prove that $A \times C \subseteq B \times D$.*

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,

$(a, b) \in A \times C \Rightarrow a \in A$ and $b \in C$

$\Rightarrow$ $a \in B$ and $b \in D$ [$\because A \subseteq B$ and $C \subseteq D$]

$\Rightarrow$ $(a, b) \in B \times D$

$\therefore$ $A \times C \subseteq B \times D$ Q.E.D.

THEOREM 7 *For any sets A, B, C, D prove that:*

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

PROOF Let (a, b) be an arbitrary element of $(A \times B) \cap (C \times D)$. Then,

$(a, b) \in (A \times B) \cap (C \times D) \Rightarrow (a, b) \in A \times B$ and $(a, b) \in C \times D$

$\Rightarrow$ $(a \in A$ and $b \in B)$ and $(a \in C$ and $b \in D)$

$\Rightarrow$ $(a \in A$ and $a \in C)$ and $(b \in B$ and $b \in D)$

$\Rightarrow$ $a \in (A \cap C)$ and $b \in B \cap D \Rightarrow (a, b) \in (A \cap C) \times (B \cap D)$

$\therefore$ $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$

Similarly, $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$

Hence, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ Q.E.D.

COROLLARY For any sets A and B, prove that $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.

THEOREM 8 *For any three sets A, B, C prove that:*

(i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

(ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$.

PROOF (i) We have,

$A \times (B' \cup C')' = A \times ((B')' \cap (C')')$ [By De-Morgan's law]

$= A \times (B \cap C) = (A \times B) \cap (A \times C)$ [See Theorem 1]

(ii) $A \times (B' \cap C')' = A \times (B')' \cup (C')'$ [By De-Morgan's Law]

$= A \times (B \cup C) = (A \times B) \cup (A \times C)$ [See Theorem 1]

Q.E.D.

THEOREM 9 *Let A and B be two non-empty sets having n elements in common, then prove that $A \times B$ and $B \times A$ have n^2 elements in common.*

PROOF We have $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ [See Theorem 7]

$\Rightarrow (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ [On replacing C by B and D by A]

$\Rightarrow (A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$

It is given that $A \cap B$ has n elements, so $(A \cap B) \times (B \cap A)$ has n^2 elements.

But, $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$

$\therefore (A \times B) \cap (B \times A)$ has n^2 elements.

Hence, $A \times B$ and $B \times A$ have n^2 elements in common. Q.E.D.

THEOREM 10 *Let A be a non-empty set such that $A \times B = A \times C$. Show that $B = C$.*

PROOF Let b be an arbitrary element of B. Then,

$(a, b) \in A \times B$ for all $a \in A$

$\Rightarrow (a, b) \in A \times C$ for all $a \in A$ [$\because A \times B = A \times C$]

$\Rightarrow b \in C$

Thus, $b \in B \Rightarrow b \in C$

$\therefore B \subset C$ ...(i)

Now, let c be an arbitrary element of C. Then,

$(a, c) \in A \times C$ for all $a \in A$

$\Rightarrow (a, c) \in A \times B$ for all $a \in A$ [$\because A \times B = A \times C$]

$\Rightarrow c \in B$

Thus, $c \in C \Rightarrow c \in B$

$\therefore C \subset B$ ...(ii)

From (i) and (ii), we get

$B = C$

EXERCISE 2.2

1. Given $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, find $(A \times B) \cap (B \times C)$.
2. If $A = \{2, 3\}$, $B = \{4, 5\}$, $C = \{5, 6\}$, find $A \times (B \cup C)$, $A \times (B \cap C)$, $(A \times B) \cup (A \times C)$.
3. If $A = \{1, 2, 3\}$, $B = \{4\}$, $C = \{5\}$, then verify that:
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (iii) $A \times (B - C) = (A \times B) - (A \times C)$.
4. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that:
   (i) $A \times C \subset B \times D$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
5. If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, find
   (i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$ (iii) $A \times (B \cup C)$ (iv) $(A \times B) \cup (A \times C)$
6. Prove that: (i) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ (ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
7. If $A \times B \subseteq C \times D$ and $A \times B \neq \phi$, prove that $A \subseteq C$ and $B \subseteq D$.

ANSWERS

1. $\{3, 4\}$.
2. $A \times (B \cup C) = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$
   $A \times (B \cap C) = \{(2, 5), (3, 5)\}$.
   $(A \times B) \cup (A \times C) = \{(2, 4), (2, 5), (3, 4), (3, 5), (2, 6), (3, 6)\}$.
4. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
5. (i) $\{(1, 4), (2, 4), (3, 4)\}$ (ii) $\{(1, 4), (2, 4), (3, 4)\}$
   (iii) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$
   (iv) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

HINTS TO SELECTED PROBLEM

6. $n(A \times B) = n(A) \cdot n(B) = 5 \times 4 = 20$. From theorem 9, *if* A and B have n elements in common, then $(A \times B)$ and $B \times A$ have n^2 elements in common. Therefore, $n[(A \times B) \cap (B \times A) = 3^2 = 9$.

2.5 RELATIONS

Let A and B denote the sets of all male and female members in the royal family of Dasrath's kingdom. Clearly, A = {Dasrath, Ram, Laxman, Shatrughan, Bharat} and B = {Kaushalya, Kaikai, Sumitra, Sita, Urmila, Shrutkirti, Mandvi}.

If we write R for the relation "was husband of" then the fact that Dasrath was husband of Kaushalya, Kaikai and Sumitra, Ram was husband of Sita, Laxman was husband of Urmila, Bharat was husband of Mandvi and Shatrughan was husband of Shrutkirti can be represented as:

Dasrath R Kaushlya, Dasrath R Kaikai, Dasrath R Sumitra, Ram R Sita, Laxman R Urmila, Bharat R Mandvi and Shatrughan R Shrutkirti.

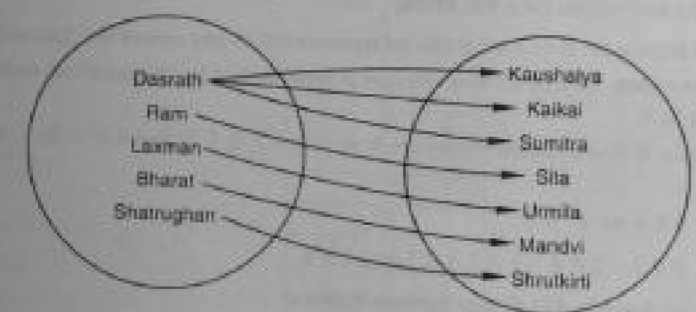


Fig. 2.10

Now, if we omit the letter R between the pairs of names and write them as ordered pairs, then the above fact can also be written as a set R of ordered pairs, where

R = {(Dasrath, Kaushalya), (Dasrath, Kaikai), (Dasrath, Sumitra), (Ram, Sita), (Laxman, Urmila), (Bharat, Mandvi), (Shatrughan, Shrutkirti)}.

Clearly, $R \subseteq A \times B$.

A visual representation of this relation R in the form of an arrow diagram is as follows:

Thus, we see that the relation "was husband of" from set A to set B gives rise to a subset R of $A \times B$ such that $(x, y) \in R$ iff xRy.

Keeping this example in mind, we may define a relation as follows.

RELATION *Let A and B be two sets. Then a relation R from A to B is a subset of A × B.*

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

If R is a relation from a non-void set A to a non-void set B and if $(a, b) \in R$, then we write aRb which is read as 'a is related to b by the relation R'. If $(a, b) \notin R$, then we write $a \not R b$ and we say that a is not related to b by the relation R.

ILLUSTRATION 1 *If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then $R = \{(1, b), (2, c), (1, a), (3, a)\}$, being a subset of $A \times B$, is a relation from A to B. Here $(1, b), (2, c), (1, a)$ and $(3, a) \in R$, so we write $1Rb, 2Rc, 1Ra$ and $3Ra$. But $(2, b) \notin R$, so we write $2 \not R b$.*

ILLUSTRATION 2 *If $A = \{a, b, c, d\}$, $B = \{p, q, r, s\}$, then which of the following are relations from A to B? Give reasons for your answer.*

(i) $R_1 = \{(a, p), (b, r), (c, s)\}$ (ii) $R_2 = \{(q, b), (c, s), (d, r)\}$

(iii) $R_3 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$ (iv) $R_4 = \{(a, p), (q, a), (b, s), (s, b)\}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, R_1 is a relation from A to B.

(ii) Since $(q, b) \in R_2$ but $(q, b) \notin A \times B$. So, $R_2 \not\subseteq A \times B$. Thus, R_2 is not a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So R_3 is a relation from A to B.

(iv) R_4 is not a relation from A to B, because (q, a) and (s, b) are elements of R_4 but (q, a) and (s, b) are not in $A \times B$. As such $R_4 \not\subseteq A \times B$.

TOTAL NUMBER OF RELATIONS Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines a relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

2.5.1 REPRESENTATION OF A RELATION

A relation from a set A to a set B can be represented in any one of the following forms:

(i) ROSTER FORM In this form a relation is represented by the set of all ordered pairs belonging to R.

For example, if R is a relation from set $A = \{-2, -1, 0, 1, 2\}$ to set $B = \{0, 1, 4, 9, 10\}$ by the rule

$$a R b \Leftrightarrow a^2 = b$$

Then, $0 R 0, -2 R 4, -1 R 1, 1 R 1$ and $2 R 4$.

So, R can be described in Roster form as follows:

$$R = \{(0, 0), (-1, 1), (-2, 4), (1, 1), (2, 4)\}$$

(ii) SET-BUILDER FORM In this form the relation R from set A to set B is represented as

$$R = \{(a, b) : a \in A, b \in B \text{ and } a, b \text{ satisfy the rule which associates } a \text{ and } b\}$$

For example, if $A = \{1, 2, 3, 4, 5\}$, $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\right\}$ and R is a relation from A to B given by

$$R = \left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$$

Then, R in set-builder form can be described as follows:

$$R = \left\{(a, b) : a \in A, b \in B \text{ and } b = \frac{1}{a}\right\}$$

It should be noted that it is not possible to express every relation from set A to set B in set-builder form. For example, the relation $R = \{(1, a), (1, c), (3, b)\}$ from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c\}$ cannot be described in set-builder form.

(iii) BY ARROW DIAGRAM In order to represent a relation from set A to a set B by an arrow diagram, we draw arrows from first components to the second components of all ordered pairs belonging to R.

For example, relation $R = \{(1, 2), (2, 4), (3, 2), (1, 3), (3, 4)\}$ from set $A = \{1, 2, 3, 4, 5\}$ to set $B = \{2, 3, 4, 5, 6, 7\}$ can be represented by the following arrow diagram:

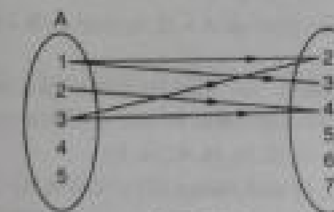


Fig. 2.11

(iv) BY LATTICE In this form, the relation R from set A to set B is represented by darkening the dots in the lattice for $A \times B$ which represent the ordered pairs in R.

For example, if $R = \{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$ is a relation from set $A = \{-3, -2, -1, 0, 1, 2, 3\}$ to set $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then R can be represented by the following lattice.

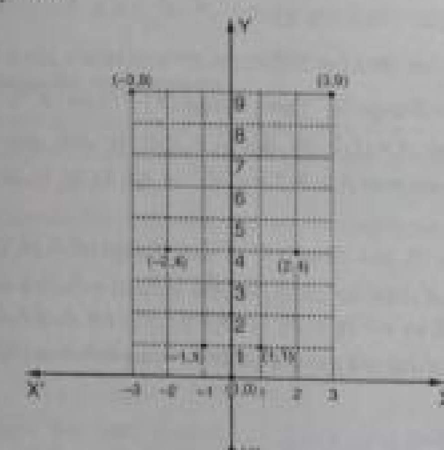


Fig. 2.12

2.5.2 DOMAIN AND RANGE OF A RELATION

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{a : (a, b) \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

The set B is called the co-domain of relation R.

ILLUSTRATION 1 *If* $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ *and let* $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ *be a relation from A to B. Then,*

$$\text{Dom}(R) = \{1, 3, 5\} \text{ and Range}(R) = \{8, 6, 2, 4\}$$

ILLUSTRATION 2 *Let* $A = \{1, 3, 5, 7\}$ *and* $B = \{2, 4, 6, 8\}$ *be two sets and let R be a relation from A to B defined by the phrase "*$(x, y) \in R \Leftrightarrow x > y$*". Under this relation R, we have*

$3R2, 5R2, 5R4, 7R2, 7R4$ and $7R6$

i.e. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$.

$\therefore$ Dom$(R) = \{3, 5, 7\}$ and Range $(R) = \{2, 4, 6\}$

ILLUSTRATION 3 *If R is a relation from set* $A = \{2, 4, 5\}$ *to set* $B = \{1, 2, 3, 4, 6, 8\}$ *defined by* $xRy \Leftrightarrow x$ *divides y.*

(i) Write R as a set of ordered pairs. (ii) Find the domain and the range of R.

SOLUTION (i) Clearly, $2R2, 2R4, 2R6, 2R8, 4R4$, and $4R8$.

$\therefore$ $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$

(ii) Clearly, Dom$(R) = \{2, 4\}$ and Range $(R) = \{2, 4, 6, 8\}$

RELATION ON A SET *Let A be a non-void set. Then, a relation from A to itself i.e. a subset of* $A \times A$*, is called a relation on set A.*

2.5.3 INVERSE OF A RELATION

INVERSE RELATION *Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by* R^{-1}*, is a relation from B to A and is defined by*

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1}).

ILLUSTRATION 1 *Let* $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ *be two sets and let* $R = \{(1, a), (1, c), (2, d), (2, c)\}$ *be a relation from A to B. Then* $R^{-1} = \{(a, 1), (c, 1), (d, 2), (c, 2)\}$ *is a relation from B to A.*

Also, Dom $(R) = \{1, 2\}$ = Range (R^{-1}), and Range $(R) = \{a, c, d\}$ = Dom (R^{-1}).

ILLUSTRATION 2 *Let A be the set of first ten natural numbers and let R be a relation on A defined by* $(x, y) \in R \Leftrightarrow x + 2y = 10$*, i.e.* $R = \{(x, y) : x \in A, y \in A \text{ and } x + 2y = 10\}$*. Express R and* R^{-1} *as sets of ordered pairs. Determine also (i) domains of R and* R^{-1} *(ii) ranges of R and* R^{-1}.

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \frac{10 - x}{2}, x, y \in A \text{ where}$$

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Now, $x = 1 \Rightarrow y = \frac{10 - 1}{2} = \frac{9}{2} \notin A$

This shows that 1 is not related to any element in A. Similarly we can observe that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation.

Further we find that :

For $x = 2$, $y = \frac{10 - 2}{2} = 4 \in A$ $\therefore$ $(2, 4) \in R$

For $x = 4$, $y = \frac{10 - 4}{2} = 3 \in A$ $\therefore$ $(4, 3) \in R$

For $x = 6$, $y = \frac{10 - 6}{2} = 2 \in A$ $\therefore$ $(6, 2) \in R$

For $x = 8$, $y = \frac{10 - 8}{2} = 1 \in A$ $\therefore$ $(8, 1) \in R$

Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\} \Rightarrow R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, Dom$(R) = \{2, 4, 6, 8\}$ = Range (R^{-1}) and Range $(R) = \{4, 3, 2, 1\}$ = Dom (R^{-1}).

ILLUSTRATIVE EXAMPLES

Type I **ON EXAMINING WHETHER A SET OF ORDERED PAIRS REPRESENTS A RELATION OR NOT**

EXAMPLE 1 *If* $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$*, which of the following are relations from A to B? Give reasons in support of your answer.*

(i) $R_1 = \{(1, 4), (1, 5), (1, 6)\}$ (ii) $R_2 = \{(1, 5), (2, 4), (3, 6)\}$

(iii) $R_3 = \{(1, 4), (1, 5), (3, 6), (2, 6), (3, 4)\}$ (iv) $R_4 = \{(4, 2), (2, 6), (5, 1), (2, 4)\}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, it is a relation from A to B.

(ii) Clearly, $R_2 \subseteq A \times B$. So, it is a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So, it is a relation from A to B.

(iv) Since $(4, 2) \in R_4$ but $(4, 2) \notin A \times B$. So, R_4 is not a relation from A to B.

Type II **ON DESCRIBING A RELATION AND ITS INVERSE AS A SET OF ORDERED PAIRS AND FINDING THEIR DOMAINS AND RANGES**

EXAMPLE 2 *A relation R is defined from a set* $A = \{2, 3, 4, 5\}$ *to a set* $B = \{3, 6, 7, 10\}$ *as follows:* $(x, y) \in R \Leftrightarrow x$ divides y*. Express R as a set of ordered pairs and determine the domain and range of R. Also, find* R^{-1}.

SOLUTION Recall that a/b stands for 'a divides b'. For the elements of the given sets A and B, we find that $2/6, 2/10, 3/3, 3/6$, and $5/10$.

$\therefore$ $(2, 6) \in R, (2, 10) \in R, (3, 3) \in R, (3, 6) \in R$ and $(5, 10) \in R$.

Thus, $R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$.

Clearly, Dom $(R) = \{2, 3, 5\}$ and Range $(R) = \{3, 6, 10\}$.

Also, $R^{-1} = \{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$.

EXAMPLE 3 *If R is the relation "less than" from* $A = \{1, 2, 3, 4, 5\}$ *to* $B = \{1, 4, 5\}$*, write down the set of ordered pairs corresponding to R. Find the inverse of R.*

SOLUTION It is given that $(x, y) \in R \Leftrightarrow x < y$, where $x \in A$ and $y \in B$.

For the elements of the given sets A and B, we find that

$1 < 4, 1 < 5, 2 < 4, 2 < 5, 3 < 4, 3 < 5$ and $4 < 5$

$\therefore$ $(1, 4) \in R, (1, 5) \in R, (2, 4) \in R, (2, 5) \in R, (3, 4) \in R, (3, 5) \in R$ and $(4, 5) \in R$.

Thus, $R = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$.

This shows that $R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$

$= \{(x, y) : x \in B, y \in A \text{ and } x > y\}$

EXAMPLE 4 A relation R is defined on the set Z of integers as follows:

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25$$

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25 \Leftrightarrow y = \pm\sqrt{25 - x^2}$$

We observe that $x = 0 \Rightarrow y = \pm 5$

$\therefore$ $(0, 5) \in R$ and $(0, -5) \in R$

$x = \pm 3 \Rightarrow y = \sqrt{25 - 9} = \pm 4$

$\therefore$ $(3, 4) \in R, (-3, 4) \in R, (3, -4) \in R$ and $(-3, -4) \in R$

$x = \pm 4 \Rightarrow y = \sqrt{25 - 16} = \pm 3$

$\therefore$ $(4, 3) \in R, (-4, 3) \in R, (4, -3) \in R$ and $(-4, -3) \in R$

$x = \pm 5 \Rightarrow y = \sqrt{25 - 25} = 0$

$\therefore$ $(5, 0) \in R$ and $(-5, 0) \in R$

We also notice that for any other integral value of x, y is not an integer

$\therefore$ $R = \{(0, 5), (0, -5), (3, 4), (-3, 4), (3, -4), (-3, -4), (4, 3), (-4, 3), (4, -3), (-4, -3), (5, 0), (-5, 0)\}$

$R^{-1} = \{(5, 0), (-5, 0), (4, 3), (4, -3), (-4, 3), (-4, -3), (3, 4), (3, -4), (-3, 4), (-3, -4), (0, 5), (0, -5)\}$

Clearly domain $(R) = \{0, 3, -3, 4, -4, 5, -5\}$ = domain (R^{-1}).

EXAMPLE 5 *Let R be the relation on the set N of natural numbers defined by $R = \{(a, b) : a + 3b = 12, a \in N, b \in N\}$. Find :*

(i) R (ii) *Domain of R* (iii) *Range of R*

SOLUTION (i) We have,

$a + 3b = 12$

$\Rightarrow$ $a = 12 - 3b$

Putting $b = 1, 2, 3$ we get $a = 9, 6, 3$ respectively.

For $b = 4$, we get $a = 0 \notin N$. Also, for $b > 4$, $a \notin N$.

$\therefore$ $R = \{(9, 1), (6, 2), (3, 3)\}$

(ii) Domain of $R = \{9, 6, 3\}$

(iii) Range of $R = \{1, 2, 3\}$

Type III ON REPRESENTING A RELATION BY USING AN ARROW DIAGRAM

EXAMPLE 6 *Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation R on set A by $R = \{(x, y) : y = x + 1\}$*

(i) *Depict this relation using an arrow diagram.*

(ii) *Write down the domain, co-domain and range of R.*

SOLUTION (i) Putting $x = 1, 2, 3, 4, 5, 6$ in $y = x + 1$, we get $y = 2, 3, 4, 5, 6, 7$.

For $x = 6$, we get $y = 7$ which does not belong to set A.

$\therefore$ $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

The arrow diagram representing R is as follows:

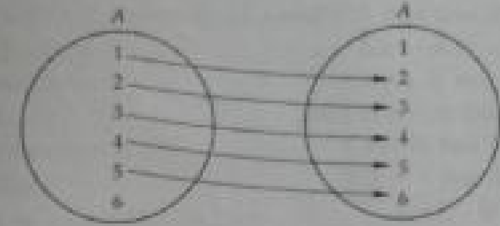


Fig. 2.13

(ii) Clearly, Domain $(R) = \{1, 2, 3, 4, 5\}$, Range $(R) = \{2, 3, 4, 5, 6\}$.

EXAMPLE 7 *The adjacent figure shows a relation R between the sets P and Q. Write this relation R in (i) Set builder form (ii) Roster form.*

What is its domain and range?

SOLUTION (i) We have,

$R = \{(9, 3), (9, -3), (4, 2), (4 - 2), (25, 5), (25 - 5)\}$

(ii) It is evident from R, that it consists of elements (x, y), where x is the square of y i.e. $x = y^2$. Therefore, relation R in Roster form is

$R = \{(x, y) : x = y^2, x \in P, y \in Q\}$

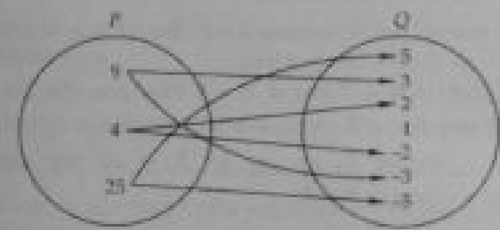


Fig. 2.14

The domain of R is $\{9, 4, 25\}$

The range of R is $\{-5, -3, -2, 2, 3, 5\}$

REMARK *In the above example, the range of relation R is not same as the set Q. The set Q is known as the co-domain.*

Type IV ON PROVING RESULTS BASED ON THE DEFINITION OF A RELATION

EXAMPLE 8 *Let R be a relation on Q defined by*

$R = \{(a, b) : a, b \in Q$ and $a - b \in Z\}$ [NCERT]

Show that:

(i) $(a, a) \in R$ for all $a \in Q$

(ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

SOLUTION (i) For any $a \in Q$, we have

$a - a = 0 \in Z$

$\Rightarrow$ $(a, a) \in R$

Hence, $(a, a) \in R$ for all $a \in Q$.

(ii) Let $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow a - b \in Z$, where $a, b \in Q$

$\Rightarrow b - a \in Z$ $\quad [\because b - a = -(a - b)]$

$\Rightarrow (b, a) \in R$

(iii) Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a - b \in Z$ and $b - c \in Z$

$\Rightarrow$ $(a - b) + (b - c) \in Z$

$\Rightarrow$ $a - c \in Z$

$\Rightarrow$ $(a, c) \in R$

EXAMPLE 9 *Let R be a relation on N defined by*

$$R = \{(a, b) : a, b \in N \text{ and } a = b^2\}$$ [NCERT]

Are the following true:

(i) $(a, a) \in R$ *for all* $a \in N$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$

Justify your answer in each case.

SOLUTION (i) We observe that $a = a^2$ is true for $a = 1 \in N$ only.

Therefore, $(1, 1) \in R$. But, $(2, 2), (3, 3), (4, 4)$ etc does not belong to R.

So, $(a, a) \in R$ for all $a \in N$ is not true.

(ii) We observe that $(4, 2) \in N$, because $4 = 2^2$. But, $(2, 4) \notin N$ as $2 \neq 4^2$.

So, $(a, b) \in R \Rightarrow (b, a) \in R$ is not true.

(iii) We observe that $(16, 4) \in R$ and $(4, 2) \in R$. However, $(16, 2) \notin R$.

So, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ is not true.

EXAMPLE 10 *Let a relation R_1 on the set R of all real numbers be defined as* $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$ *for all* $a, b \in R$.

Show that:

(i) $(a, a) \in R_1$ *for all* $a \in R$

(ii) $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ *for all* $a, b \in R$

SOLUTION (i) For any $a \in R$, we have

$1 + a^2 > 0$

$\Rightarrow$ $(a, a) \in R_1$

Thus, $(a, a) \in R_1$ for all $a \in R$.

(ii) Let $(a, b) \in R_1$. Then,

$(a, b) \in R_1 \Rightarrow 1 + ab > 0$

$\Rightarrow 1 + ba > 0$

$\Rightarrow (b, a) \in R_1$

Thus, $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$.

EXAMPLE 11 *Let R be the relation on the set Z of all integers defined by*

$$(x, y) \in R \Rightarrow x - y \text{ is divisible by } n$$

Prove that:

(i) $(x, x) \in R$ *for all* $x \in Z$

(ii) $(x, y) \in R \Rightarrow (y, x) \in R$ *for all* $x, y \in Z$

(iii) $(x, y) \in R$ *and* $(y, z) \in R \Rightarrow (x, z) \in R$ *for all* $x, y, z \in R$.

SOLUTION (i) For any $x \in Z$, we have

$x - x = 0 = 0 \times n$

$\Rightarrow$ $x - x$ is divisible by n

$\Rightarrow$ $(x, x) \in R$

Thus, $(x, x) \in R$ for all $x \in Z$.

(ii) Let $(x, y) \in R$. Then,

$(x, y) \in R \Rightarrow x - y$ is divisible by n

$\Rightarrow x - y = \lambda n$ for some $\lambda \in Z$

$\Rightarrow y - x = (-\lambda) n$

$\Rightarrow y - x$ is divisible by n $\quad [\because \lambda \in Z \Rightarrow -\lambda \in Z]$

$\Rightarrow (y, x) \in R$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$.

(iii) Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R \Rightarrow x - y$ is divisible by $n \Rightarrow x - y = \lambda n$ for some $\lambda \in Z$

$(y, z) \in R \Rightarrow y - z$ is divisible by $n \Rightarrow y - z = \mu n$ for some $\mu \in Z$

$\therefore$ $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ $x - y = \lambda n$ and $y - z = \mu n$

$\Rightarrow$ $(x - y) + (y - z) = \lambda n + \mu n$

$\Rightarrow$ $x - z = (\lambda + \mu) n$

$\Rightarrow$ $x - z$ is divisible by n $\quad [\because \lambda + \mu \in Z]$

$\Rightarrow$ $(x, z) \in R$

Thus, $(x, y) \in Z$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

EXERCISE 2.3

1. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following are relations from A to B ? Give reasons in support of your answer.
   (i) $\{(1, 6), (3, 4), (5, 2)\}$ (ii) $\{(1, 5), (2, 6), (3, 4), (3, 6)\}$
   (iii) $\{(4, 2), (4, 3), (5, 1)\}$ (iv) $A \times B$.
2. A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows:
   $$(x, y) \in R \Leftrightarrow x \text{ is relatively prime to } y$$
   Express R as a set of ordered pairs and determine its domain and range.
3. Let A be the set of first five natural numbers and let R be a relation on A defined as follows:
   $$(x, y) \in R \Leftrightarrow x \leq y$$
   Express R and R^{-1} as sets of ordered pairs. Determine also (i) the domain of R^{-1} (ii) the range of R.
4. Find the inverse relation R^{-1} in each of the following cases:
   (i) $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$
   (ii) $R = \{(x, y) : x, y \in N, x + 2y = 8\}$
   (iii) R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$.
5. Write the following relations as the sets of ordered pairs:
   (i) A relation R from the set $\{2, 3, 4, 5, 6\}$ to the set $\{1, 2, 3\}$ defined by $x = 2y$.
   (ii) A relation R on the set $\{1, 2, 3, 4, 5, 6, 7\}$ defined by $(x, y) \in R \Leftrightarrow x$ is relatively prime to y.

(iii) A relation R on the set $\{0, 1, 2, ..., 10\}$ defined by $2x + 3y = 12$.

(iv) A relation R from a set $A = \{5, 6, 7, 8\}$ to the set $B = \{10, 12, 15, 16, 18\}$ defined by $(x, y) \in R \Leftrightarrow x$ divides y.

6. Let R be a relation in N defined by $(x, y) \in R \Leftrightarrow x + 2y = 8$. Express R and R^{-1} as sets of ordered pairs.

7. Let $A = \{3, 5\}$ and $B = \{7, 11\}$. Let $R = \{(a, b) : a \in A, b \in B, a - b$ is odd$\}$. Show that R is an empty relation from A into B.

8. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the total number of relations from A into B.

9. Determine the domain and range of the relation R defined by

(i) $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$ [NCERT]

(ii) $R = \{(x, x^3) : x$ is a prime number less than $10\}$ [NCERT]

10. Determine the domain and range of the following relations:

(i) $R = \{(a, b) : a \in N, a < 5, b = 4\}$

(ii) $S = \{(a, b) : b = |a - 1|, a \in Z$ and $|a| \le 3\}$

11. Let $A = \{a, b\}$. List all relations on A and find their number.

12. Let $A = \{x, y, z\}$ and $B = \{a, b\}$. Find the total number of relations from A into B. [NCERT]

13. Let R be a relation from N to N defined by $R = \{(a, b) : a, b \in N$ and $a = b^2\}$. Are the following statements true?

(i) $(a, a) \in R$ for all $a \in N$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

14. Let $A = \{1, 2, 3, ..., 14\}$. Define a relation on a set A by

$R = \{(x, y) : 3x - y = 0$, where $x, y \in A\}$.

Depict this relationship using an arrow diagram. Write down its domain, co-domain and range.

15. Define a relation R on the set N of natural numbers by $R = \{(x, y) : y = x + 5$, x is a natural number less than 4, $x, y \in N\}$. Depict this relationship using (i) roster form (ii) an arrow diagram. Write down the domain and range or R. [NCERT]

16. $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. Define a relation R from A to B by $R = \{(x, y)$: the difference between x and y is odd, $x \in A$, $y \in B\}$. Write R in Roster form. [NCERT]

17. Write the relation $R = \{(x, x^3) : x$ is a prime number less than $10\}$ in roster form. [NCERT]

18. Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be a relation on A defined by $\{(a, b) : a, b \in A$, b is exactly divisible by $a\}$

(i) Write R in roster form (ii) Find the domain of R [NCERT]

(iii) Find the range of R.

19. The adjacent figure shows a relationship between the sets P and Q. Write this relation in (i) set builder form (ii) roster form. What is its domain and range? [NCERT]

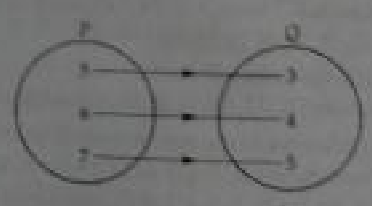


Fig. 2.15

20. Let R be the relation on Z defined by

$R = \{(a, b) : a, b \in Z, a - b$ is an integer$\}$ [NCERT]

Find the domain and range of R.

21. For the relation R_1 defined on R by the rule $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$.

Prove that: $(a, b) \in R_1$ and $(b, c) \in R_1 \Rightarrow (a, c) \in R_1$ is not true for all $a, b, c \in R$.

22. Let R be a relation on $N \times N$ defined by

$(a, b) R (c, d) \Leftrightarrow a + d = b + c$ for all $(a, b), (c, d) \in N \times N$

Show that:

(i) $(a, b) R (a, b)$ for all $(a, b) \in N \times N$

(ii) $(a, b) R (c, d) \Rightarrow (c, d) R (a, b)$ for all $(a, b), (c, d) \in N \times N$

(iii) $(a, b) R (c, d)$ and $(c, d) R (e, f) \Rightarrow (a, b) R (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$

ANSWERS

1. (i) It is not a relation from A to B.

(ii) It is a subset of $A \times B$, so it is a relation from A to B.

(iii) It is not a relation from A to B as it is not a subset of $A \times B$.

(iv) It is a relation from A to B.

2. $R = \{(2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 7)\}$

3. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5)\}$

$R^{-1} = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (2, 2), (3, 2), (4, 2), (5, 2), (3, 3), (4, 3), (5, 3), (4, 4), (5, 4), (5, 5)\}$

Domain of $R^{-1} = \{1, 2, 3, 4, 5\}$ = Range of R.

4. (i) $R^{-1} = \{(2, 1), (3, 1), (3, 2), (2, 3), (6, 5)\}$

(ii) $R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$

(iii) $R^{-1} = \{(8, 11), (10, 13)\}$

5. (i) $\{(2, 1), (4, 2), (6, 3)\}$

(ii) $\{(2, 3), (2, 5), (2, 7), (3, 2), (3, 4), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7), (5, 2), (5, 3), (5, 4), (5, 6), (5, 7), (6, 5), (6, 7), (7, 2), (7, 3), (7, 4), (7, 5), (7, 6), (7, 7)\}$

(iii) $\{(0, 4), (3, 2), (6, 0)\}$

(iv) $\{(5, 10), (5, 15), (6, 12), (6, 18), (8, 16)\}$

6. $R = \{(2, 3), (4, 2), (6, 1)\}$ $R^{-1} = \{(3, 2), (2, 4), (1, 6)\}$ 8. 16

9. (i) Domain $R = \{0, 1, 2, 3, 4, 5\}$, Range $R = \{5, 6, 7, 8, 9, 10\}$

(ii) Domain $R = \{2, 3, 5, 7\}$, Range $R = \{8, 27, 125, 343\}$

10. (i) Domain $R = \{1, 2, 3, 4\}$, Range $R = \{4\}$

Domain $S = \{0, -1, -2, -3, 1, 2, 3\}$, Range $S = \{0, 1, 2, 3, 4\}$

(ii) $S = \{(0, 1)), (-1, 2), (-2, 3), (-3, 4), (1, 0), (2, 1), (3, 2)\}$

11. 16 12. 64 13. (i) No (ii) No (iii) No

14.

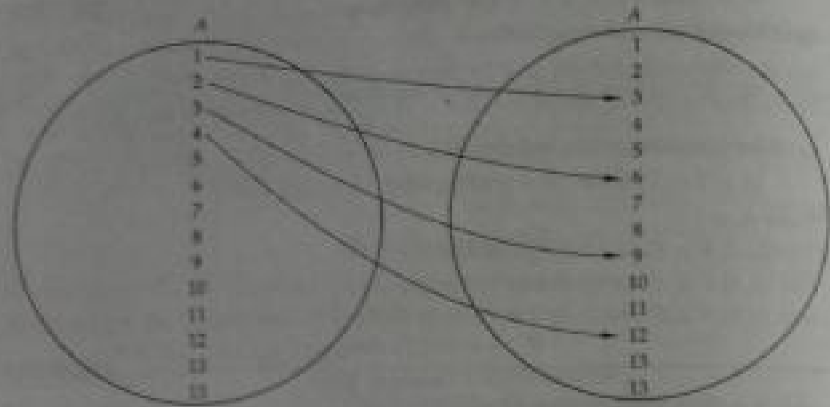


Fig. 2.16

Domain $(R) = \{1, 2, 3, 4\}$, Co-domain $(R) = A$, Range $(R) = \{3, 6, 9, 12\}$

15. (i) $R = \{(1, 6), (2, 7), (3, 8)\}$
(ii) Domain $(R) = \{1, 2, 3\}$, Range $(R) = \{6, 7, 8\}$

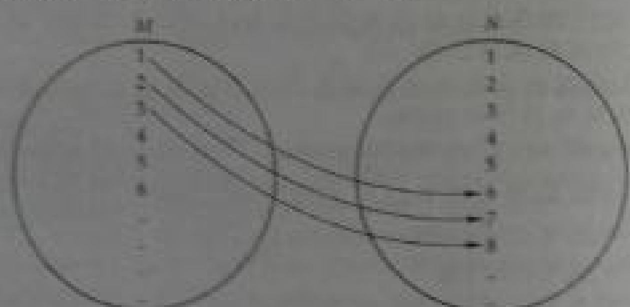


Fig. 2.17

16. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$
17. $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$
18. (i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$
(ii) Domain $(R) = \{1, 2, 3, 4, 5, 6\}$
(iii) Range $(R) = \{1, 2, 3, 4, 5, 6\}$
19. (i) $R = \{(x, y) : y = x - 2, x \in P, y \in Q\}$ (ii) $R = \{(5, 3), (6, 4), (7, 5)\}$
Domain $(R) = \{5, 6, 7\}$, Range $(R) = \{3, 4, 5\}$
20. (i) Domain $(R) = Z$, Range $(R) = Z$

HINTS TO NCERT & SELECTED PROBLEMS

8. We have, $n(A) = 2$, $n(B) = 2$
$\therefore \quad n(A \times B) = 2 \times 2 = 4$
So, there are $2^4 = 16$ relations from A to B.

9. (i) We have,
$R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\} = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$
$\therefore$ Domain $(R) = \{0, 1, 2, 3, 4, 5\}$ and, Range $(R) = \{5, 6, 7, 8, 9, 10\}$
(ii) We have,
$R = \{(x, x^3) : x$ is a prime number less than $10\} = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$
$\therefore$ Domain $(R) = \{2, 3, 5, 7\}$, Range $(R) = \{8, 27, 125, 343\}$

10. (i) We have,
$R = \{(1, 4), (2, 4), (3, 4), (4, 4)\}$
$\therefore$ Domain $(R) = \{1, 2, 3, 4\}$, Range $(R) = \{4\}$
(ii) We have,
$S = \{(-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1), (3, 2)\}$
$\therefore$ Domain $(S) = \{-3, -2, -1, 0, 1, 2, 3\}$, Range $(S) = \{0, 1, 2, 3, 4\}$

12. Here A has 3 elements and B has 2 elements. Therefore, total number of relations from A to $B = 2^{3 \times 2} = 64$.

13. (i) No, because $(2, 2) \notin R$.
(ii) No, because $(4, 2) \in R$ but $(2, 4) \notin R$.
(iii) No, because $(16, 4) \in R$ and $(4, 2) \in R$ but $(16, 2) \notin R$.

14. $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$
Domain $(R) = \{1, 2, 3, 4\}$, Range $(R) = \{3, 6, 9, 12\}$.

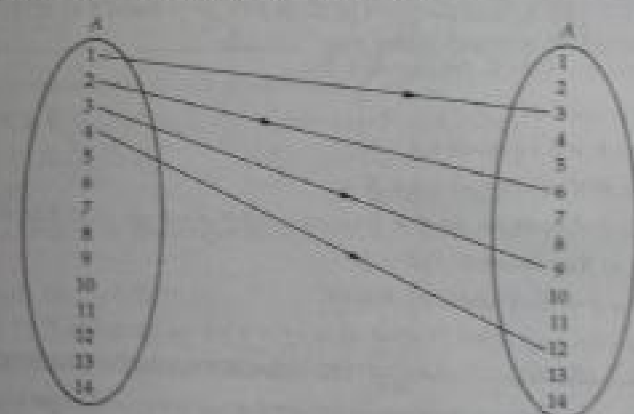


Fig. 2.18

15. (i) We have,
$R = \{(x, y) : y = x + 5, x$ is a natural number less than $4, x, y \in N\} = \{(1, 6), (2, 7), (3, 8)\}$
(ii)

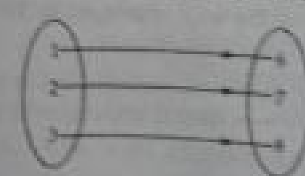


Fig. 2.19

(ii) Domanin $(R) = \{1, 2, 3\}$, Range $(R) = \{6, 7, 8\}$

16. We have, $A = \{1, 2, 3, 5\}, B = \{4, 6, 9\}$

$R = \{(x, y)$: the difference between x and y is odd, $x \in A, y \in B\}$

$\therefore \quad R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

17. We have,

$R = \{(x, x^3) : x$ is a prime number less than $10\}$

$\Rightarrow \quad R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

18. (i) We have,

$R = \{(a, b) : a, b \in A, b$ is exactly divisible by $a\}$, where $A = \{1, 2, 3, 4, 5, 6\}$.

$\therefore R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

(i) Domain $R = \{1, 2, 3, 4, 5, 6\}$ (ii) Range $R = \{1, 2, 3, 4, 5, 6\}$

19. (i) $\{(x, y) : y = x - 2, x \in \{5, 6, 7\}, y \in \{3, 4, 5\}\}$

(ii) $\{(5, 3), (6, 4), (7, 5)\}$

Domain $R = \{5, 6, 7\}$, Range $R = \{3, 4, 5\}$

20. The relation R on Z is defined by $R = \{(a, b) : a, b \in Z, a - b$ is an integer$\}$

Since $a - b$ is an integer for all $a, b \in Z$. So, domain $(R) = Z =$ Range (R).

21. We have,

$\left(1, -\frac{1}{2}\right) \in R_1$ and $\left(-\frac{1}{2}, -4\right) \in R_1$ as $1 + \left(\frac{-1}{2}\right) > 0$ and $1 + \left(\frac{-1}{2}\right)(-4) > 0$

But, $1 + 1 \times -4 \not> 0$. So, $(1, -4) \in R_1$.

22. (i) We know that

$a + b = b + a$ for all $a, b \in N$

$\therefore \quad (a, b)\, R\, (a, b)$ for all $a, b \in N$

(ii) $(a, b)\, R\, (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a \Rightarrow (c, d)\, R\, (a, b)$

(iii) $(a, b)\, R\, (c, d)$ and $(c, d)\, R\, (e, f)$

$\Rightarrow \quad a + d = b + c$ and $c + f = d + e$

$\Rightarrow \quad a + d + c + f = b + c + d + e \Rightarrow a + f = b + e \Rightarrow (a, b)\, R\, (e, f)$

VE! Y SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $A = \{1, 2, 4\}, B = \{2, 4, 5\}$ and $C = \{2, 5\}$, write $(A - C) \times (B - C)$.
2. If $n(A) = 3, n(B) = 4$, then write $n(A \times A \times B)$.
3. If R is a relation defined on the set Z of integers by the rule $(x, y) \in R \Leftrightarrow x^2 + y^2 = 9$, then write domain of R.
4. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation defined on the set Z of integers, then write domain of R.
5. If R is a relation from set $A = \{11, 12, 13\}$ to set $B = \{8, 10, 12\}$ defined by $y = x - 3$, then write R^{-1}.

6. Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : |a^2 - b^2| \leq 5, a, b \in A\}$. Then write R as set of ordered pairs.
7. Let $R = \{(x, y) : x, y \in Z, y = 2x - 4\}$. If $(a, -2)$ and $(4, b^2) \in R$, then write the values of a and b.
8. If $R = \{(2, 1), (4, 7), (1, -2), \ldots\}$, then write the linear relation between the components of the ordered pairs of the relation R.
9. If $A = \{1, 3, 5\}$ and $B = \{2, 4\}$, list the elements of R, if $R = \{(x, y) : x, y \in A \times B$ and $x > y\}$
10. If $R = \{(x, y) : x, y \in W, 2x + y = 8\}$, then write the domain and range of R.
11. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, write A and B.
12. Let $A = \{1, 2, 3, 5\}$, $B = \{4, 6, 9\}$ and R be a relation from A to B defined by $R = \{(x, y) : x - y$ is odd$\}$. Write R in roster form.

ANSWERS

1. $\{(1, 4), (4, 4)\}$
2. 36
3. Domain $(R) = \{-3, 0, 3\}$
4. Domain $(R) = \{-2, -1, 0, 1, 2\}$
5. $\{(8, 11), (10, 13)\}$
6. $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$
7. $a = 1, b = \pm 2$
8. $y = 3x - 5$
9. $\{(3, 2), (5, 2), (5, 4)\}$
10. Domain $(R) = \{0, 1, 2, 3, 4\}$, Range $(R) = \{0, 2, 4, 6, 8\}$
11. $A = \{x, y, z\}, B = \{1, 2\}$
12. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $A = \{1, 2, 4\}, B = \{2, 4, 5\}, C = \{2, 5\}$, then $(A - B) \times (B - C)$ is

(a) $\{(1, 2), (1, 5), (2, 5)\}$ (b) $\{(1, 4)\}$

(c) $(1, 4)$ (d) none of these.

2. If R is a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ given by $x\, R\, y \Leftrightarrow y = 3x$, then $R =$

(a) $\{(3, 1), (6, 2), (8, 2), (9, 3)\}$ (b) $\{(3, 1), (6, 2), (9, 3)\}$

(c) $\{(3, 1), (2, 6), (3, 9)\}$ (d) none of these.

3. Let $A = \{1, 2, 3\}, B = \{1, 3, 5\}$. If relation R from A to B is given by $R = \{(1, 3), (2, 5), (3, 3)\}$. Then, R^{-1} is

(a) $\{(3, 3), (3, 1), (5, 2)\}$ (b) $\{(1, 3), (2, 5), (3, 3)\}$

(c) $\{(1, 3), (5, 2)\}$ (d) none of these.

4. If $A = \{1, 2, 3\}, B = \{1, 4, 6, 9\}$ and R is a relation from A to B defined by 'x is greater than y. The range of R is

(a) $\{1, 4, 6, 9\}$ (b) $\{4, 6, 9\}$

(c) $\{1\}$ (d) none of these.

5. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation on Z, then domain of R is

(a) $\{0, 1, 2\}$ (b) $\{0, -1, -2\}$

(c) $\{-2, -1, 0, 1, 2\}$ (d) none of these.

6. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by : $x R y \Leftrightarrow x$ is relatively prime to y. Then, domain of R is

(a) $\{2, 3, 5\}$ (b) $\{3, 5\}$

(c) $\{2, 3, 4\}$ (d) $\{2, 3, 4, 5\}$

7. A relation ϕ from C to R is defined by $x \phi y \Leftrightarrow |x| = y$. Which one is correct?

(a) $(2 + 3i) \phi 13$ (b) $3 \phi (-3)$

(c) $(1 + i) \phi 2$ (d) $i \phi 1$.

8. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is

(a) $\{2, 4, 8\}$ (b) $\{2, 4, 6, 8\}$

(c) $\{2, 4, 6\}$ (d) $\{1, 2, 3, 4\}$

9. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then, R^{-1} is

(a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 8), (13, 10)\}$

(c) $\{(10, 13), (8, 11), (12, 10)\}$ (d) none of these.

10. If the set A has p elements, B has q elements, then the number of elements in $A \times B$ is

(a) $p + q$ (b) $p + q + 1$

(c) pq (d) p^2

11. Let R be a relation from a set A to a set B, then

(a) $R = A \cup B$ (b) $R = A \cap B$

(c) $R \subseteq A \times B$ (d) $R \subseteq B \times A$.

12. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is

(a) 2^{mn} (b) $2^{mn} - 1$ (c) $2mn$ (d) m^n

13. If R is a relation on a finite set having n elements, then the number of relations on A is

(a) 2^n (b) 2^{n^2} (c) n^2 (d) n^n.

ANSWERS

1. (b) 2. (d) 3. (a) 4. (c) 5. (c) 6. (d) 7. (d) 8. (c)
9. (a) 10. (c) 11. (c) 12. (a) 13. (b)

SUMMARY

1. An ordered pair consists of two objects or elements in a given fixed order.
2. $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$
3. If A and B are two non-empty sets, then
$A \times B = \{(a, b) : a \in A, b \in B\}$ is called the cartesian product of A and B.
If A and B are finite sets having m and n elements respectively, then $A \times B$ has mn elements.
4. $R \times R = \{(x, y) : x, y \in R\}$ is the set of all points in xy-plane.
5. $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ set of all points in three dimensional space.

6. For any three sets A, B, C, we have
(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
(iii) $A \times (B - C) = A \times B - A \times C$
(iv) $A \times B = B \times A \Leftrightarrow A = B$
(v) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
(vi) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
(vii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$
(viii) $A \times B = A \times C \Rightarrow B = C$
7. Let A and B be two sets. A relation from A to B is a subset of $A \times B$.
8. If A and B are finite sets having m and n elements respectively. Then, 2^{mn} relations can be defined from A to B.
9. If R is a relation from set A to set B, then
Domain $(R) = \{x : (x, y) \in R\}$, Range $(R) = \{y : (x, y) \in R\}$
10. A relation from a set A to itself is called a relation on A.
11. Let A, B be two sets and let R be a relation from set A to set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by
$R^{-1} = \{(b, a) : (a, b) \in R\}$
Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$
Domain (R) = Range (R^{-1}), Range (R) = Domain (R^{-1}).

3

FUNCTIONS

3.1 INTRODUCTION

In this chapter, we shall study about one of the most important concepts in mathematics known as a function. Functions form one of the most important building blocks of Mathematics. The word "Function" is derived from a Latin word meaning operation and the words mapping and map are synonymous to it. Functions play a very important role in differential and integral calculus which you will study in XII class. In this chapter, we shall introduce the concept of a function as a correspondence between two sets. We shall also study function as a relation from one set to the other set.

3.2 FUNCTION AS A SPECIAL KIND OF RELATION

DEFINITION *Let A and B be two non-empty sets. A relation f from A to B, i.e., a sub-set of $A \times B$, is called a function (or a mapping or a map) from A to B, if*

(i) *for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$*

(ii) *$(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.*

Thus, a non-void subset f of $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$, then b is called the image of a under f.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and f_1, f_2 and f_3 be three subsets of $A \times B$ as given below:*

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},$$
$$f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\},$$
$$f_3 = \{(1, 3), (2, 4)\}.$$

Then, f_1 is a function from A to B but f_2 and f_3 are not functions from A to B. f_2 is not a function from A to B, because $1 \in A$ has two images 2 and 3 in B and f_3 is not a function from A to B because $3 \in A$ has no image in B.

If a function f is expressed as the set of ordered pairs, the domain of f is the set of all first components of members of f and the range of f is the set of second components of members of f i.e.,

$$\text{Domain of } f = \{a : (a, b) \in f\}, \quad \text{Range of } f = \{b : (a, b) \in f\}$$

ILLUSTRATION 2 *If $x, y \in \{1, 2, 3, 4\}$, then which of the following are functions in the given set?*

(a) $f_1 = \{(x, y) : y = x + 1\}$ (b) $f_2 = \{(x, y) : x + y > 4\}$

(c) $f_3 = \{(x, y) : y < x\}$ (d) $f_4 = \{(x, y) : x + y = 5\}$

Also, in case of a function give its range.

SOLUTION If we express f_1, f_2, f_3 and f_4 as sets of ordered pairs, then we have

$$f_1 = \{(1, 2), (2, 3), (3, 4)\},$$

$f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$,

$f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$ and $f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$.

(a) We have, $f_1 = \{(1, 2), (2, 3), (3, 4)\}$.

We observe that an element 4 of the given set has not appeared in first place of any ordered pair of f_1. So, f_1 is not a function from the given set to itself.

(b) We have, $f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$.

We observe that 2, 3, 4 have appeared more than once as first components of the ordered pairs in f_2. So, f_2 is not a function.

(c) We have, $f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$.

We observe that 3 and 4 have appeared more than once as first components of the ordered pairs in f_3. So, f_3 is not a function.

(d) We have, $f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$.

We observe that each element of the given set has appeared as first component in one and only one ordered pair of f_4. So, f_4 is a function in the given set.

In this case, Range of $f = \{1, 2, 3, 4\}$

ILLUSTRATION 3 *Let f be a relation on the set N of natural numbers defined by $f = \{(n, 3n) : n \in N\}$. Is f a function from N to N. If so, find the range of f.*

SOLUTION Since for each $n \in N$, there exists a unique $3n \in N$ such that $(n, 3n) \in f$. Therefore, f is a function from N to N.

Now,

$$\text{Range of } f = \{f(n) : n \in N\} = \{3n : n \in N\}.$$

ILLUSTRATION 4 *Let f be a subset of $Z \times Z$ defined by $f = \{(ab, (a + b)) : a, b \in Z\}$. Is f a function from Z into Z. Justify your answers.* [NCERT]

SOLUTION We observe that :

$1 \times 6 = 6$ and $2 \times 3 = 6$

$\Rightarrow$ $(1 \times 6, 1 + 6) \in f$ and $(2 \times 3, 2 + 3) \in f$

$\Rightarrow$ $(6, 7) \in f$ and $(6, 5) \in f$

So, f is not a function from Z to Z.

3.3 FUNCTION AS A CORRESPONDENCE

DEFINITION *Let A and B be two non-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that:*

(i) *all elements of set A are associated to elements in set B.*

(ii) *an element of set A is associated to a unique element in set B.*

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B.

Terms such as "map" (or "mapping"), "correspondence" are used as synonyms for "function". If f is a function from a set A to a set B, then we write $f : A \rightarrow B$ or $A \xrightarrow{f} B$, which is read as *f is a function from A to B or f maps A to B.*

If an element $a \in A$ is associated to an element $b \in B$, then b is called '*the f-image of a*' or '*image of a under f*' or '*the value of the function f at a*'. Also, a is called the pre-image of b under the function f. We write it as : $b = f(a)$.

ILLUSTRATION *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements of A to elements of B as shown in the following figures:*

We observe that f_1 is not a function from set A to set B, since there is an element $3 \in A$ which is not associated to any element of B.

Also, f_2 is not a function from A to B because an element $4 \in A$ is associated to two elements c and e in B. But, f_3 and f_4 are functions from A to B, because under f_3 and f_4 each element in A is associated to a unique element in B.

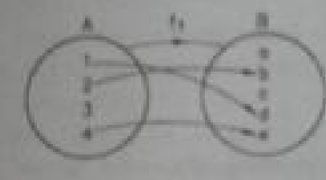


Fig. 3.1

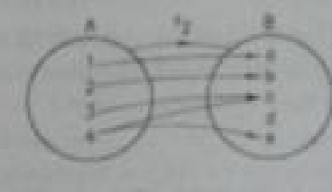


Fig. 3.2

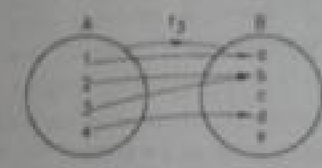


Fig. 3.3

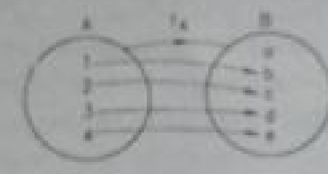


Fig. 3.4

3.4 DESCRIPTION OF A FUNCTION

Let $f : A \rightarrow B$ be a function such that the set A consists of a finite number of elements. Then, $f(x)$ be described by listing the values which it attains at different points of its domain. For example, if $A = \{-1, 1, 2, 3\}$ and B is the set of real numbers, then a function $f : A \rightarrow B$ can be described as $f(-1) = 3$, $f(1) = 0$, $f(2) = 3/2$ and $f(3) = 0$. In case, A is an infinite set, then f cannot be described by listing the images at points in its domain. In such cases functions are generally described by a formula. For example, $f : Z \rightarrow Z$ given by $f(x) = x^2 + 1$ or $f : R \rightarrow R$ given by $f(x) = e^x$ etc.

3.5 DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let $f : A \rightarrow B$. Then, the set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

Thus, $f(A) = \{f(x) : x \in A\}$ = Range of f

Clearly, $f(A) \subseteq B$.

ILLUSTRATION 1 *Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4, 5, 6\}$.*

Consider a rule $f(x) = x^2$.

Then, $f(-2) = (-2)^2 = 4$, $f(-1) = (-1)^2 = 1$, $f(0) = 0^2 = 0$, $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$.

Clearly, each element of A is associated to a unique element of B. So, $f : A \rightarrow B$ given by $f(x) = x^2$ is a function.

Clearly, domain $(f) = A = \{-2, -1, 0, 1, 2\}$ and range $(f) = \{0, 1, 4\}$.

ILLUSTRATION 2 *Consider a rule $f(x) = 2x - 3$ associating elements of N (set of natural numbers) to elements of N. This rule does not define a function from N to itself, because $f(1) = 2 \times 1 - 3 = -1 \notin N$ i.e. $1 \in N$ is not associated to any element of N.*

ILLUSTRATION 3 *Let* $A = \{-2, -1, 0, 1, 2\}$ *and* $f : A \to Z$ *given by* $f(x) = x^2 - 2x - 3$.
Find (a) *the range of* f (b) *pre-images of* 6, −3 *and* 5.

SOLUTION

(a) We have,

$$f(-2) = (-2)^2 - 2(-2) - 3 = 5,$$
$$f(-1) = (-1)^2 - 2(-1) - 3 = 0, f(0) = -3,$$
$$f(1) = 1^2 - 2 \times 1 - 3 = -4 \text{ and } f(2) = 2^2 - 2 \times 2 - 3 = -3$$

So, range $(f) = \{0, 5, -3, -4\}$

(b) Let x be the pre-image of 6. Then,

$$f(x) = 6 \Rightarrow x^2 - 2x - 3 = 6 \Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = 1 \pm \sqrt{10}$$

Since $x = 1 \pm \sqrt{10} \notin A$. So, there is no pre-image of 6.

Let x be the pre-image of −3. Then,

$$f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2.$$

Clearly, $0, 2 \in A$. So, 0 and 2 are pre-images of − 3.

Let x be the pre-image of 5. Then,

$$f(x) = 5 \Rightarrow x^2 - 2x - 3 = 5$$
$$\Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x - 4)(x + 2) = 0 \Rightarrow x = 4, -2$$

Since, $-2 \in A$. So, −2 is the pre-image of 5.

3.6 EQUAL FUNCTION

DEFINITION *Two functions* f *and* g *are said to be equal iff*

(i) *domain of* f = *domain of* g,

(ii) *co-domain of* f = *co-domain of* g,

and (iii) $f(x) = g(x)$ *for every* x *belonging to their common domain.*

If two functions f and g are equal, then we write $f = g$.

ILLUSTRATION 1 *Let* $A = \{1, 2\}$, $B = \{3, 6\}$ *and* $f : A \to B$ *given by* $f(x) = x^2 + 2$ *and* $g : A \to B$ *given by* $g(x) = 3x$. *Then, we observe that* f *and* g *have the same domain and co-domain. also we have,* $f(1) = 3 = g(1)$ *and* $f(2) = 6 = g(2)$

Hence, $f = g$.

ILLUSTRATION 2 *Let* $f : R - \{2\} \to R$ *be defined by* $f(x) = \dfrac{x^2 - 4}{x - 2}$ *and* $g : R \to R$ *be defined by* $g(x) = x + 2$. *Find whether* $f = g$ *or not.*

SOLUTION Clearly, $f(x) = g(x)$ for all $x \in R - \{2\}$.
But, $f(x)$ and $g(x)$ have different domains.
Infact, domain of $f = R - \{2\}$ and domain of $g = R$. Therefore, $f \neq g$.

ILLUSTRATION 3 *Let* $f : Z \to Z$, $g : Z \to Z$ *be functions defined by* $f = \{(n, n^2) : n \in Z\}$
$g = \{(n, |n|^2) : n \in Z\}$. *Show that* $f = g$.

SOLUTION Clearly, domain of f = Domain of $g = Z$

and Co-domain of f = Co-domain of $g = Z$

We have, $f(n) = n^2$ and $g(n) = |n|^2 = n^2$

∴ $f(n) = g(n)$ for all $n \in Z$. $[\because |n|^2 = n^2]$

Hence, $f = g$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Express the following functions as sets of ordered pairs and determine their ranges*

(a) $f : A \to R$, $f(x) = x^2 + 1$, *where* $A = \{-1, 0, 2, 4\}$.

(b) $g : A \to N$, $g(x) = 2x$, *where* $A = \{x : x \in N, x \le 10\}$.

SOLUTION

(a) We have,

$$f(-1) = (-1)^2 + 1 = 2,$$
$$f(0) = 0^2 + 1 = 1, f(2) = 2^2 + 1 = 5 \text{ and } f(4) = 4^2 + 1 = 17$$
$$\therefore \quad f = \{(x, f(x)) : x \in A\} = \{(-1, 2), (0, 1), (2, 5), (4, 17)\}$$

Hence, Range of $(f) = \{2, 1, 5, 17\}$

(b) We have,

$A = \{1, 2, 3, \ldots\ldots 10\}$. Therefore,

$g(1) = 2 \times 1 = 2$, $g(2) = 2 \times 2 = 4$, $g(3) = 2 \times 3 = 6$, $g(4) = 2 \times 4 = 8$,
$g(5) = 2 \times 5 = 10$, $g(6) = 2 \times 6 = 12$, $g(7) = 2 \times 7 = 14$, $g(8) = 2 \times 8 = 16$,
$g(9) = 2 \times 9 = 18$ and $g(10) = 2 \times 10 = 20$.

∴ $g = \{(x, g(x)) : x \in A\} = \{(1, 2), (2, 4), (3, 6), \ldots, (10, 20)\}$.

Range of $g = g(A) = \{g(x) : x \in A\} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.

EXAMPLE 2 *Find the domain for which the functions* $f(x) = 2x^2 - 1$ *and* $g(x) = 1 - 3x$ *are equal.*

SOLUTION We have,

$$f(x) = g(x)$$
$$\Rightarrow \quad 2x^2 - 1 = 1 - 3x$$
$$\Rightarrow \quad 2x^2 + 3x - 2 = 0 \Rightarrow (x + 2)(2x - 1) = 0 \Rightarrow x = -2, 1/2.$$

Thus, $f(x)$ and $g(x)$ are equal on the set $\{-2, 1/2\}$.

EXAMPLE 3 *Is* $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ *a function ? If this is described by the formula,* $g(x) = \alpha x + \beta$, *then what values should be assigned to* α *and* β ?

SOLUTION Since no two ordered pairs in g have the same first component. So, g is a function such that $g(1) = 1$, $g(2) = 3$, $g(3) = 5$ and $g(4) = 7$.

It is given that $g(x) = \alpha x + \beta$.

∴ $g(1) = 1$ and $g(2) = 3 \Rightarrow \alpha + \beta = 1$ and $2\alpha + \beta = 3 \Rightarrow \alpha = 2, \beta = -1$.

EXAMPLE 4 *Given* $A = \{-1, 0, 2, 5, 6, 11\}$, $B = \{-2, -1, 0, 18, 28, 108\}$ *and* $f(x) = x^2 - x - 2$. *Is* $f(A) = B$? *Find* $f(A)$.

SOLUTION We have,

$$f(-1) = (-1)^2 - (-1) - 2 = 0,\ f(0) = 0^2 - 0 - 2 = -2,\ f(2) = 2^2 - 2 - 2 = 0,$$
$$f(5) = 5^2 - 5 - 2 = 18,\ f(6) = 6^2 - 6 - 2 = 28 \text{ and } f(11) = 11^2 - 11 - 2 = 108.$$
$$\therefore \quad f(A) = \{f(x) : x \in A\}$$
$$= \{f(-1), f(0), f(2), f(5), f(6), f(11)\}$$
$$= \{0, -2, 18, 28, 108\}$$

We observe that $-1 \in B$, but $-1 \notin f(A)$. So, $f(A) \neq B$.

EXAMPLE 5 *Let* $f : R \to R$ *be given by* $f(x) = x^2 + 3$. *Find* (a) $\{x : f(x) = 28\}$ (b) *the pre-images of* 39 *and* 2 *under* f.

SOLUTION We have,

(a) $f(x) = 28 \Rightarrow x^2 + 3 = 28 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$

∴ $\{x : f(x) = 28\} = \{-5, 5\}$.

(b) Let x be the pre-image of 39. Then,

$$f(x)=39 \Rightarrow x^2+3=39 \Rightarrow x^2=36 \Rightarrow x=\pm 6$$

So, pre-images of 39 are -6 and 6.

Let x be the pre-image of 2. Then,

$$f(x)=2 \Rightarrow x^2+3=2 \Rightarrow x^2=-1$$

Since, no real value of x satisfies the equation $x^2=-1$. Therefore, 2 does not have any pre-image under f.

EXAMPLE 6 *Let $f: R \to R$ be a function given by $f(x)=x^2+1$. Find:*

(i) $f^{-1}\{-5\}$ (ii) $f^{-1}\{26\}$ (iii) $f^{-1}\{10, 37\}$

SOLUTION Recall that if $f: A \to B$ such that $y \in B$. Then, $f^{-1}\{y\}=\{x \in A : f(x)=y\}$. In other words, $f^{-1}\{y\}$ is the set of pre-images of y.

(i) Let $f^{-1}\{-5\}=x$. Then, $f(x)=-5 \Rightarrow x^2+1=-5$

Clearly, this equation is not solvable in R. So, $f^{-1}\{-5\}=\phi$

(ii) Let $f^{-1}\{26\}=x$. Then, $f(x)=26 \Rightarrow x^2+1=26 \Rightarrow x=\pm 5$

$\therefore$ $f^{-1}\{26\}=\{-5, 5\}$.

(iii) Let $f^{-1}\{10, 37\}=x$. Then, $f(x)=10$ or $f(x)=37$

$\therefore$ $x^2+1=10$ or $x^2+1=37 \Rightarrow x=\pm 3$ or $x=\pm 6$

So, $f^{-1}\{10, 37\}=\{3, -3, 6, -6\}$.

EXAMPLE 7 *Let $f=\{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function described by the formula $f(x)=ax+b$ for some integers a, b. Determine a, b.* [NCERT]

SOLUTION We have,

$$f(1)=1, f(2)=3, f(0)=-1 \text{ and } f(-1)=-3.$$

It is given that $f(x)=ax+b$. Therefore,

$f(1)=1$ and $f(2)=3$

$\Rightarrow$ $a+b=1$ and $2a+b=3 \Rightarrow a=2, b=-1$.

Thus, $f(x)=2x-1$. Clearly, $f(0)=-1$ and $f(-1)=-3$ are true.

Hence, $a=2$ and $b=-1$.

EXAMPLE 8 *If $f: R \to R$ be defined as follows:*

$$f(x)=\begin{cases} 1, & \text{if } x \in Q \\ -1, & \text{if } x \notin Q \end{cases}$$

Find (a) $f(1/2), f(\pi), f(\sqrt{2})$ (b) *Range of f* (c) *pre-images of 1 and -1.*

SOLUTION

(a) It is evident from the definition of f that at every rational point the function attains value 1 and at every irrational point it attains value -1.

$\therefore$ $\frac{1}{2} \in Q \Rightarrow f\left(\frac{1}{2}\right)=1, \pi \notin Q \Rightarrow f(\pi)=-1$ and $\sqrt{2} \notin Q \Rightarrow f(\sqrt{2})=-1$.

(b) We have, Range of $f=\{f(x) : x \in R\}$.

Also, by definition $f(x)$ attains values 1 or -1 according as x is rational or irrational and a real number is either rational or irrational.

$\therefore$ Range of $f=\{1, -1\}$.

(c) Since $f(x)=1$ for all $x \in Q$. Therefore, pre-images of 1 are rational numbers i.e.

$$f^{-1}(1)=Q.$$

Also, -1 is the image of every real number which is not rational.

$\therefore$ $f^{-1}(-1)=R-Q=$ Set of irrational numbers.

EXAMPLE 9 *Let $f: R \to R$ be such that $f(x)=2^x$. Determine:*

(a) *Range of f* (b) $\{x : f(x)=1\}$ (c) *whether $f(x+y)=f(x) \cdot f(y)$ holds.*

SOLUTION

(a) Since 2^x is positive for every $x \in R$. So, $f(x)=2^x$ is a positive real number for every $x \in R$. Moreover, for every positive real number x, there exist $\log_2 x \in R$ such that

$$f(\log_2 x)=2^{\log_2 x}=x \qquad [\because a^{\log_a x}=x]$$

Hence, we conclude that the range of f is the set of all positive real numbers.

(b) $\because$ $f(x)=1 \Rightarrow 2^x=1 \Rightarrow 2^x=2^0 \Rightarrow x=0$.

$\therefore$ $\{x : f(x)=1\}=\{0\}$.

(c) We have,

$f(x+y)=2^{x+y}=2^x \cdot 2^y=f(x) \cdot f(y)$

$\therefore$ $f(x+y)=f(x) f(y)$ holds for all $x, y \in R$.

EXAMPLE 10 *Let A be the set of two positive integers. Let $f: A \to Z^+$ (set of positive integers) be defined by*

$f(n)=p$, where p is the highest prime factor of n.

If range of $f=\{3\}$. Find set A. Is A uniquely determined?

SOLUTION It is given that the set A consists of two positive integers. So, let $A=\{n, m\}$.

Since range of $f=\{3\}$.

$\therefore$ $f(n)=3$ and $f(m)=3$

$\Rightarrow$ Highest prime factors of n and m both are equal to 3.

$\Rightarrow$ ($n=3$ and $m=6$) or ($n=3$ and $m=9$) or ($n=3$ and $m=12$)

or ($n=6$ and $m=12$) etc.

$\Rightarrow$ $A=\{3, 6\}$ or $A=\{3, 9\}$ or $A=\{3, 12\}$, or $A=\{6, 12\}$ etc.

Clearly, A is not uniquely determined.

EXAMPLE 11 *Let $A \subseteq N$ and $f: A \to A$ be defined by*

$f(n)$: the highest prime factor of n.

If range of f is A. Determine A. Is A uniquely determined?

SOLUTION For any $n \in A$, we have

$f(n)=$ Highest prime factor of n

$\Rightarrow$ Range of f consists of prime number only

But, it is given that range of f is A. Therefore, set A consists of prime numbers only.

Hence, $A=$ set of some prime numbers.

Clearly, A is not uniquely determined.

EXERCISE 3.1

1. Define a function as a set of ordered pairs.
2. Define a function as a correspondence between two sets.
3. What is the fundamental difference between a relation and a function? Is every relation a function?
4. Let $A=\{-2, -1, 0, 1, 2\}$ and $f: A \to Z$ be a function defined by $f(x)=x^2-2x-3$. Find:
   (a) range of f i.e. $f(A)$ (b) pre-images of 6, -3 and 5.

5. If a function $f : R \to R$ be defined by

$$f(x) = \begin{cases} 3x - 2, & x < 0; \\ 1, & x = 0; \\ 4x + 1, & x > 0. \end{cases}$$

Find: $f(1), f(-1), f(0), f(2)$.

6. A function $f : R \to R$ is defined by $f(x) = x^2$. Determine (a) range of f, (b) $\{x : f(x) = 4\}$, (c) $\{y : f(y) = -1\}$.

7. Let $f : R^+ \to R$, where R^+ is the set of all positive real numbers, be such that $f(x) = \log_e x$. Determine
   (a) the image set of the domain of f
   (b) $\{x : f(x) = -2\}$
   (c) whether $f(xy) = f(x) + f(y)$ holds.

8. Write the following relations as sets of ordered pairs and find which of them are functions:
   (a) $\{(x, y) : y = 3x, x \in \{1, 2, 3\}, y \in \{3, 6, 9, 12\}\}$.
   (b) $\{(x, y) : y > x + 1, x = 1, 2 \text{ and } y = 2, 4, 6\}$
   (c) $\{(x, y) : x + y = 3, x, y \in \{0, 1, 2, 3\}\}$

9. Let $f : R \to R$ and $g : C \to C$ be two functions defined as $f(x) = x^2$ and $g(x) = x^2$. Are they equal functions?

10. If f, g, h are three functions defined from R to R as follows:
   (i) $f(x) = x^2$ (ii) $g(x) = \sin x$ (iii) $h(x) = x^2 + 1$
   Find the range of each function.

11. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 5, 9, 11, 15, 16\}$
   Determine which of the following sets are functions from X to Y
   (a) $f_1 = \{(1, 1), (2, 11), (3, 1), (4, 15)\}$
   (b) $f_2 = \{(1, 1), (2, 7), (3, 5)\}$
   (c) $f_3 = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$. [NCERT]

12. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f : A \to Z$ be a function given by
   $f(x)$ = highest prime factor of x.
   Find range of f.

13. If $f : R \to R$ be defined by $f(x) = x^2 + 1$, then find $f^{-1}\{17\}$ and $f^{-1}\{-3\}$.

14. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?
   (a) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$ (b) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$
   (c) $R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$ (d) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$.

15. Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \to N$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f. [NCERT]

16. The function f is defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$
   The relation g is defined by $g(x) = \begin{cases} x^2, & 0 \le x \le 2 \\ 3x, & 2 \le x \le 10 \end{cases}$ [NCERT]
   Show that f is a function and g is not a function.

17. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1) - 1}$ [NCERT]

ANSWERS

4. (a) $f(A) = \{-4, -3, 0, 5\}$ (b) ϕ, $\{0, 2\}$, -2
5. $f(1) = 5$, $f(-1) = -5$, $f(0) = 1$, $f(2) = 9$.
6. (a) R^+ (set of all real numbers greater than or equal to zero) (b) $\{-2, 2\}$ (c) ϕ
7. (a) R (b) $\{e^{-2}\}$ (c) Yes
8. (a) $\{(1, 3), (2, 6), (3, 9)\}$; Function (b) $\{(1, 4), (1, 6), (2, 4), (2, 6)\}$; Not a function
   (c) $\{(0, 3), (1, 2), (2, 1), (3, 0)\}$; Function.
9. No, since domain of $f \neq$ domain of g.
10. (i) $R^+ = \{x \in R \mid x \ge 0\}$ (ii) $\{x \in R : -1 \le x \le 1\}$ (iii) $\{x \in R : x \ge 1\}$.
11. (a) 12. $\{3, 13, 7, 5, 2, 17\}$
13. $f^{-1}(17) = \{-4, 4\}$, $f^{-1}(-3) = \phi$ 14. (c) 15. Range $(f) = \{3, 5, 11, 13\}$. 17. 2.1

HINTS TO NCERT & SELECTED PROBLEMS

11. (c) $f_3 = \{(1,5), (2, 9), (3, 1), (4, 5), (2, 11)\}$ is not a function because $(2, 9)$ and $(2, 11) \in f_3$
12. Clearly, $f(12)$ = highest prime factor of $12 = 3$. Similarly, $f(13) = 13$, $f(14) = 7$, $f(15) = 5$, $f(16) = 2$ and $f(17) = 17$
15. We have,
   $A = \{9, 10, 11, 12, 13\}$ and $f : A \to N$ is defined by $f(n)$ = the highest prime factor of n.
   $\therefore$ $f(9) = 3$, $f(10) = 5$, $f(11) = 11$, $f(12) = 3$ and $f(13) = 13$
   Hence, range $(f) = \{3, 5, 11, 13\}$
16. We observe that $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 3x, & 3 \le x \le 10 \end{cases}$ associates all numbers in $[0, 10]$ to numbers in R and no number in $[0, 10]$ is associated to two or more numbers. Hence, f is a function. But, g is not a function because 2 is associated to two distinct elements viz. 4 and 6.
17. We have, $f(x) = x^2$

$$\therefore \frac{f(1.1) - f(1)}{(1.1) - 1} = \frac{(1.1)^2 - 1^2}{(1.1) - 1} = \frac{(1.1 + 1)(1.1 - 1)}{(1.1 - 1)} = 2.1$$

3.7 REAL FUNCTIONS

In this section, we will discuss functions having domain and co-domain both as subsets of the set R of all real numbers. Such functions are called real functions or real valued functions of the real variable as defined below.

REAL VALUED FUNCTION *A function $f : A \to B$ is called a real valued function, if B is a subset of R (set of all real numbers).*

If A and B both are subsets of R, then f is called a real function.

In section 3.4, we have discussed the description of a function. Generally domain and co-domain both are infinite subsets of R in case of real functions of real variable. Therefore, a real function is generally described by some general formula. In other words, images of various elements in the domain of a real function are provided by some general formula. For example, $f : R \to R$ given by $f(x) = x^2 + x + 1$ or, $f : A \to B$ given by $f(x) = \dfrac{x - 1}{x^2 - 4}$ etc.

In practice, real functions are described by giving the general expression or formula describing it without mentioning its domain and co-domain.
Following are some examples of real functions.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x-1)$*

SOLUTION We have,

$$f(x) = 3x^4 - 5x^2 + 9$$

$$\therefore \quad f(x-1) = 3(x-1)^4 - 5(x-1)^2 + 9 = 3x^4 - 12x^3 + 13x^2 - 2x + 7$$

EXAMPLE 2 *If $f(x) = x + \frac{1}{x}$, prove that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.*

SOLUTION We have,

$$f(x) = x + 1/x$$

$$\therefore \quad f(x^3) = x^3 + \frac{1}{x^3} \text{ and } [f(x)]^3 = \left(x + \frac{1}{x}\right)^3$$

Now, $[f(x)^3] = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = f(x^3) + 3f(x)$ and $f(x) = f\left(\frac{1}{x}\right)$

$$\therefore \quad [f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$$

EXAMPLE 3 *If $f(x) = \frac{1}{2x+1}$, $x \neq -\frac{1}{2}$, then show that $f(f(x)) = \frac{2x+1}{2x+3}$, provided that $x \neq -\frac{3}{2}$.*

SOLUTION We have,

$$f(x) = \frac{1}{2x+1}$$

$$\Rightarrow \quad f(f(x)) = f\left(\frac{1}{2x+1}\right) = \frac{1}{2\left(\frac{1}{2x+1}\right)+1}$$

$$\Rightarrow \quad f(f(x)) = \frac{1}{\frac{2}{2x+1}+1}$$

$$\Rightarrow \quad f(f(x)) = \frac{2x+1}{2+2x+1}$$

$$\Rightarrow \quad f(f(x)) = \frac{2x+1}{2x+3}$$

Clearly, $f(f(x)) = \frac{2x+1}{2x+3}$ is real for $2x+3 \neq 0$ i.e. $f(f(x))$ is defined for $2x+3 \neq 0$

i.e. $\quad x \neq -\frac{3}{2}$

Hence, $f(f(x)) = \frac{2x+1}{2x+3}$

EXAMPLE 4 *If $f(x) = \frac{x-1}{x+1}$, $x \neq -1$, then show that $f(f(x)) = -\frac{1}{x}$, provided that $x \neq 0$.*

SOLUTION We have,

$$f(x) = \frac{x-1}{x+1}, x \neq -1$$

$$\Rightarrow \quad f(f(x)) = f\left(\frac{x-1}{x+1}\right)$$

$$\Rightarrow \quad f(f(x)) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}$$

$$\Rightarrow \quad f(f(x)) = \frac{x-1-x-1}{x-1+x+1}$$

$$\Rightarrow \quad f(f(x)) = \frac{-2}{2x} = -\frac{1}{x}$$

Also, $-\frac{1}{x}$ is meaningful for $x \neq 0$.

Hence, $f(f(x)) = -\frac{1}{x}$, provided that $x \neq 0$.

EXAMPLE 5 *Let f be defined by $f(x) = x - 4$ and g be defined by*

$$g(x) = \begin{cases} \frac{x^2-16}{x+4}, & x \neq -4 \\ \lambda, & x = -4 \end{cases}$$

Find λ such that $f(x) = g(x)$ for all x.

SOLUTION We have,

$$f(x) = g(x) \quad \text{for all } x \in R$$

$$\Rightarrow \quad f(-4) = g(-4)$$

$$\Rightarrow \quad -4 - 4 = \lambda$$

$$\Rightarrow \quad \lambda = -8$$

EXAMPLE 6 *If f is a real function defined by $f(x) = \frac{x-1}{x+1}$, then prove that: $f(2x) = \frac{3f(x)+1}{f(x)+3}$*

SOLUTION We have,

$$f(x) = \frac{x-1}{x+1}$$

$$\Rightarrow \quad \frac{f(x)+1}{f(x)-1} = \frac{x-1+x+1}{x-1-x-1} \quad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad x = \frac{f(x)+1}{1-f(x)}$$

Now, $\quad f(2x) = \frac{2x-1}{2x+1}$

$$\Rightarrow \quad f(2x) = \frac{2\left\{\frac{f(x)+1}{1-f(x)}\right\}-1}{2\left\{\frac{f(x)+1}{1-f(x)}\right\}+1}$$

$$\Rightarrow \quad f(2x) = \frac{2f(x)+2-1+f(x)}{2f(x)+2+1-f(x)}$$

$$\Rightarrow \quad f(2x) = \frac{3f(x)+1}{f(x)+3}$$

EXERCISE 3.2

1. If $f(x) = x^2 - 3x + 4$, then find the values of x satisfying the equation $f(x) = f(2x + 1)$.
2. If $f(x) = (x - a)^2 (x - b)^2$, find $f(a + b)$.
3. If $y = f(x) = \frac{ax - b}{bx - a}$, show that $x = f(y)$.
4. If $f(x) = \frac{1}{1-x}$, show that $f[f\{f(x)\}] = x$.
5. If $f(x) = (a - x^n)^{1/n}$, $a > 0$ and $n \in N$, then prove that $f(f(x)) = x$ for all x.
6. If $f(x) = \begin{cases} x^2, \text{ when } x < 0 \\ x, \text{ when } 0 \le x < 1 \\ \frac{1}{x}, \text{ when } x \ge 1 \end{cases}$ [NCERT]

   Find: (a) $f(1/2)$, (b) $f(-2)$, (c) $f(1)$, (d) $f(\sqrt{3})$ and (e) $f(\sqrt{-3})$.
7. If $f(x) = x^3 - \frac{1}{x^3}$, show that $f(x) + f\left(\frac{1}{x}\right) = 0$.
8. If $f(x) = \frac{2x}{1+x^2}$, show that $f(\tan\theta) = \sin 2\theta$.
9. If for non-zero x, $a f(x) + b f\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \ne b$, then find $f(x)$.
10. If $f(x) = \frac{x+1}{x-1}$, show that $f[f\{(x)\}] = x$.

ANSWERS

1. $x = -1, 2/3$ 2. $a^2 b^2$

6. (a) $\frac{1}{2}$ (b) 4 (c) 1 (d) $\frac{1}{\sqrt{3}}$ (e) does not exist

9. $\frac{1}{a^2 - b^2}\left\{\frac{a}{x} - bx\right\} - \frac{5}{a+b}$

HINTS TO NCERT & SELECTED PROBLEMS

6. We have,

$$f(x) = \begin{cases} x^2, & \text{when } x < 0 \\ x, & \text{when } a \le x < 1 \\ \frac{1}{x}, & \text{when } x \ge 1 \end{cases}$$

(a) $f\left(\frac{1}{2}\right) = \frac{1}{2}$ [Using $f(x) = x, 0 \le x < 1$]

(b) $f(-2) = (-2)^2 = 4$

(c) $f(1) = \frac{1}{1} = 1$

(d) $f(\sqrt{3}) = \frac{1}{\sqrt{3}}$

(e) $f(-\sqrt{3}) = (-\sqrt{3})^2 = 3$

9. We have,

$$a f(x) + b f\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad \text{...(i)}$$

$$\Rightarrow a f\left(\frac{1}{x}\right) + b f(x) = x - 5 \quad \text{...(ii)}$$

Applying (i) + (ii), we obtain

$$\left\{f(x) + f\left(\frac{1}{x}\right)\right\}(a+b) = x + \frac{1}{x} - 10 \Rightarrow f(x) + f\left(\frac{1}{x}\right) = \frac{1}{a+b}\left\{x + \frac{1}{x} - 10\right\} \quad \text{...(iii)}$$

Similarly, by applying (i)–(ii), we get

$$f(x) - f\left(\frac{1}{x}\right) = \frac{1}{a-b}\left\{\frac{1}{x} - x\right\}. \quad \text{...(iv)}$$

Adding (iii) and (iv), we obtain the value of $f(x)$.

3.8 DOMAIN OF REAL FUNCTIONS

Mathematically to define a function one has to provide its domain, co-domain and the images of elements in its domain either by giving a general formula or by listing them one by one. As the domain and co-domain of real functions are subsets of R. Therefore, conventionally, real functions are described by providing the general formula for finding the images of elements in its domain. In such cases, the domain of the real function $f(x)$ is the set of all those real numbers for which the expression for $f(x)$ or the formula for $f(x)$ assumes real values only. In otherwords, the domain of $f(x)$ is the set of all those real numbers for which $f(x)$ is meaningful. For example, a real function $f(x)$ described by the general formula $f(x) = \frac{3x-2}{x^2-1}$ assumes real values for all $x \in R$ except for $x = \pm 1$, because denominator of $\frac{3x-2}{x^2-1}$ becomes zero for $x = \pm 1$. So, domain of $f(x)$ is the set of all real numbers other than -1 and 1 i.e. domain $(f) = R - \{-1, 1\}$.

Following examples will illustrate the procedure for finding the domain of a real function of a real variable.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the domain of each of the following real valued functions:*

(i) $f(x) = \frac{1}{x+2}$ (ii) $f(x) = \frac{x-1}{x-3}$

(iii) $f(x) = \frac{2x-3}{x^2-3x+2}$ (iv) $f(x) = \frac{x^2+3x+5}{x^2-5x+4}$ [NCERT]

SOLUTION (i) We have,

$$f(x) = \frac{1}{x+2}$$

Clearly, $f(x)$ assumes real values for all real values for all x except for the values of x satisfying $x + 2 = 0$ i.e. $x = -2$.

Hence, Domain $(f) = R - \{-2\}$.

(ii) We have,

$$f(x)=\frac{x-1}{x-3}$$

We observe that $f(x)$ is a rational function of x as $\frac{x-1}{x-3}$ is a rational expression. Clearly, $f(x)$ assumes real values for all x except for the values of x for which $x-3=0$ i.e. $x=3$.

Hence, Domain $(f)=R-\{3\}$.

(iii) We have,

$$f(x)=\frac{2x-3}{x^2-3x+2}$$

Clearly, $f(x)$ is a rational function of x as $\frac{2x-3}{x^2-3x+2}$ is a rational expression.

Clearly, $f(x)$ assumes real values for all x except for all those values of x for which $x^2-3x+2=0$ i.e. $x=1, 2$.

Hence, Domain $(f)=R-\{1, 2\}$.

(iv) We have,

$$f(x)=\frac{x^2+3x+5}{x^2-5x+4}$$

Clearly, $f(x)$ is a rational function of x as $\frac{x^2+3x+5}{x^2-5x+4}$ is a rational expression in x. We observe that $f(x)$ assumes real values for all x except for all those values of x for which $x^2-5x+4=0$ i.e., $x=1, 4$.

$\therefore$ Domain $(f)=R-\{1, 4\}$.

EXAMPLE 2 *Find the domain of each of the following functions:*

(i) $f(x)=\sqrt{x-2}$ (ii) $f(x)=\frac{1}{\sqrt{1-x}}$ (iii) $f(x)=\sqrt{4-x^2}$

SOLUTION (i) We have,

$$f(x)=\sqrt{x-2}$$

Clearly, $f(x)$ assumes real values, if

$x-2\geq 0 \Rightarrow x\geq 2 \Rightarrow x\in[2, \infty)$.

Hence, Domain $(f)=[2, \infty)$

(ii) We have,

$$f(x)=\frac{1}{\sqrt{1-x}}$$

Clearly, $f(x)$ assumes real values, if

$1-x>0 \Rightarrow 1>x \Rightarrow x<1 \Rightarrow x\in(-\infty, 1)$.

Hence, Domain $(f)=(-\infty, 1)$.

(iii) We have,

$$f(x)=\sqrt{4-x^2}$$

Clearly, $f(x)$ assumes real values, if

$4-x^2\geq 0 \Rightarrow -(x^2-4)\geq 0 \Rightarrow x^2-4\leq 0 \Rightarrow (x-2)(x+2)\leq 0 \Rightarrow x\in[-2, 2]$

Hence, Domain $(f)=[-2, 2]$

EXAMPLE 3 *Find the domain of the function $f(x)$ defined by $f(x)=\sqrt{4-x}+\frac{1}{\sqrt{x^2-1}}$.*

SOLUTION Clearly, $f(x)$ is defined for

$4-x\geq 0$ and $x^2-1>0$

$\Rightarrow$ $x-4\leq 0$ and $(x-1)(x+1)>0$

$\Rightarrow$ $x\leq 4$ and $(x<-1$ or $x>1)$

$\Rightarrow$ $x\in(-\infty, -1)\cup(1, 4]$.

Hence, Domain $(f)=(-\infty, -1)\cup(1, 4]$.

3.9 RANGE OF REAL FUNCTIONS

The range of a real function of a real variable is the set of all real values taken by $f(x)$ at points in its domain. In order to find the range of a real function $f(x)$, we may use the following algorithm.

ALGORITHM

STEP I *Put $y=f(x)$*

STEP II *Solve the equation $y=f(x)$ for x in terms of y. Let $x=\phi(y)$.*

STEP III *Find the values of y for which the values of x, obtained from $x=\phi(y)$, are real and in the domain of f.*

STEP IV *The set of values of y obtained in step III is the range of f.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the domain and range of the function $f(x)$ given by $f(x)=\frac{x-2}{3-x}$*

SOLUTION We have,

$$f(x)=\frac{x-2}{3-x}$$

Domain of f: Clearly, $f(x)$ is defined for all x satisfying $3-x\neq 0$ i.e. $x\neq 3$.

Hence, Domain $(f)=R-\{3\}$.

Range of f: Let $y=f(x)$, i.e.

$$y=\frac{x-2}{3-x}$$

$\Rightarrow$ $3y-xy=x-2$

$\Rightarrow$ $x(y+1)=3y+2$

$\Rightarrow$ $x=\frac{3y+2}{y+1}$

Clearly, x assumes real values for all y except $y+1=0$ i.e. $y=-1$.

Hence, Range $(f)=R-\{-1\}$.

EXAMPLE 2 *Find the range of each of the following functions:*

(i) $f(x)=\frac{1}{x-5}$ (ii) $f(x)=\sqrt{16-x^2}$

(iii) $f(x)=\frac{x}{1+x^2}$ (iv) $f(x)=\frac{3}{2-x^2}$

SOLUTION (i) We have, $f(x) = \frac{1}{\sqrt{x-5}}$

Clearly, $f(x)$ takes real values, if

$x - 5 > 0 \Rightarrow x > 5 \Rightarrow x \in (5, \infty)$

$\therefore$ Domain $(f) = (5, \infty)$

For $x > 5$, we have

$x - 5 > 0 \Rightarrow \sqrt{x-5} > 0 \Rightarrow \frac{1}{\sqrt{x-5}} > 0 \Rightarrow f(x) > 0.$

Thus, $f(x)$ takes all real values greater than zero.

Hence, Range $(f) = (0, \infty)$.

(ii) We have, $f(x) = \sqrt{16 - x^2}$

We observe that $f(x)$ is defined for all x satisfying

$16 - x^2 \geq 0$

$\Rightarrow x^2 - 16 \leq 0$

$\Rightarrow (x-4)(x+4) \leq 0$

$\Rightarrow -4 \leq x \leq 4$

$\Rightarrow x \in [-4, 4]$

Hence, Domain $(f) = [-4, 4]$.

Let $y = f(x)$. Then,

$y = \sqrt{16 - x^2}$

$\Rightarrow y^2 = 16 - x^2$

$\Rightarrow x^2 = 16 - y^2$

$\Rightarrow x = \sqrt{16 - y^2}$

Clearly, x will take real values, if

$16 - y^2 \geq 0$

$\Rightarrow y^2 - 16 \leq 0$

$\Rightarrow (y-4)(y+4) \leq 0$

$\Rightarrow -4 \leq y \leq 4 \Rightarrow y \in [-4, 4]$

Also, $y = \sqrt{16 - x^2} \geq 0$ for all $x \in [-4, 4]$

$\therefore$ $y \in [0, 4]$ for all $x \in [-4, 4]$

Hence, Range $(f) = [0, 4]$

(iii) We have, $f(x) = \frac{x}{1+x^2}$

Clearly, domain $(f) = R$.

Let $y = f(x)$. Then,

$$y = f(x) \Rightarrow y = \frac{x}{1+x^2} \Rightarrow x^2 y - x + y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1 - 4y^2}}{2y}$$

Clearly, x will assume real values, if

$1 - 4y^2 \geq 0$ and $y \neq 0$

$\Rightarrow 4y^2 - 1 \leq 0$ and $y \neq 0$

$\Rightarrow y^2 - \frac{1}{4} \leq 0$ and $y \neq 0$

$\Rightarrow \left(y - \frac{1}{2}\right)\left(y + \frac{1}{2}\right) \leq 0$ and $y \neq 0$

$\Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$ and $y \neq 0$

$\Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right] - \{0\}$

Also, $y = 0$ for $x = 0$.

Hence, Range $(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$

(iv) We have,

$f(x) = \frac{3}{2 - x^2}$

For $f(x)$ to be real, we must have

$2 - x^2 \neq 0 \Rightarrow x \neq \pm\sqrt{2}$.

$\therefore$ Domain $(f) = R - \{-\sqrt{2}, \sqrt{2}\}$

Let $f(x) = y$. Then,

$y = f(x)$

$\Rightarrow y = \frac{3}{2 - x^2}$

$\Rightarrow 2y - x^2 y = 3$

$\Rightarrow x^2 y = 2y - 3$

$\Rightarrow x = \pm\sqrt{\frac{2y-3}{y}}$

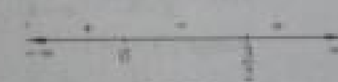


Fig. 3.5

Now, x will take real values other than $-\sqrt{2}$ and $\sqrt{2}$, if

$\frac{2y-3}{y} \geq 0 \Rightarrow y \in (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$

Hence, range $(f) = (-\infty, 0) \cup [3/2, \infty)$.

EXAMPLE 3 *Find the domain and range of the function* $f(x) = \frac{x^2 - 9}{x - 3}$

SOLUTION We have,

$f(x) = \frac{x^2 - 9}{x - 3}$

Domain of f: Clearly, $f(x)$ is not defined for $x-3=0$ i.e. $x=3$.

$\therefore$ Domain $(f) = R-\{3\}$

Range of f: Let $f(x)=y$. Then,

$$f(x) = y$$

$$\Rightarrow \quad \frac{x^2-9}{x-3} = y$$

$$\Rightarrow \quad x+3 = y \qquad [\because x \neq 3]$$

It follows from the above relation that y takes all real values except 6 when x takes values in the set $R-\{3\}$.

$\therefore$ Range $(f) = R-\{6\}$.

EXAMPLE 4 *Find the domain and range of the real valued function $f(x)$ given by $f(x) = \frac{4-x}{x-4}$*

SOLUTION We have, $f(x) = \frac{4-x}{x-4}$

Domain of f: We observe that $f(x)$ is defined for all x except at $x=4$. At $x=4$, $f(x)$ takes the indeterminate form $\frac{0}{0}$.

$\therefore$ Domain $(f) = R-\{4\}$.

Range of f: For any $x \in$ Domain (f), we have

$$f(x) = \frac{4-x}{x-4} = \frac{-(x-4)}{x-4} = -1.$$

$\therefore$ Range $(f) = \{-1\}$.

EXAMPLE 5 *Let $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ be a function from R into R. Determine the range of f.* [NCERT]

SOLUTION We have,

$$f(x) = \frac{x^2}{x^2+1}$$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ as $x^2+1 \neq 0$ for any $x \in R$.

So, Domain $(f) = R$.

Range of f: Let $f(x)=y$. Then,

$$f(x) = y$$

$$\Rightarrow \quad \frac{x^2}{x^2+1} = y$$

$$\Rightarrow \quad x^2 = x^2y + y$$

$$\Rightarrow \quad x^2(1-y) = y$$

$$\Rightarrow \quad x^2 = \frac{y}{1-y}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{y}{1-y}}$$

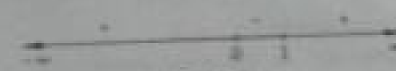


Fig. 3.6

Clearly, x will take real values, if

$$\frac{y}{1-y} \geq 0$$

$$\Rightarrow \quad \frac{y-0}{y-1} \leq 0$$

$$\Rightarrow \quad 0 \leq y < 1$$

$$\Rightarrow \quad y \in [0, 1)$$

Hence, range $(f) = [0, 1)$.

EXAMPLE 6 *Find the domain and range of the function $f = \left\{\left(x, \frac{1}{1-x^2}\right) : x \in R, x \neq \pm 1\right\}$.*

SOLUTION We have,

$$f(x) = \frac{1}{1-x^2}$$

Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ except for which $x^2-1 \neq 0$ i.e., $x=\pm 1$.

Hence, Domain of $f = R-\{-1, 1\}$.

Range of f: Let $f(x)=y$. Then,

Fig. 3.7

$$f(x) = y$$

$$\Rightarrow \quad \frac{1}{1-x^2} = y$$

$$\Rightarrow \quad 1-x^2 = \frac{1}{y}$$

$$\Rightarrow \quad x^2 = 1-\frac{1}{y} = \frac{y-1}{y}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{y-1}{y-0}}$$

Clearly, x will take real values, if

$$\frac{y-1}{y-0} \geq 0$$

$$\Rightarrow \quad y < 0 \text{ or } y \geq 1$$

$$\Rightarrow \quad y \in (-\infty, 0) \cup [1, \infty)$$

Hence, range $(f) = (-\infty, 0) \cup [1, \infty)$

EXAMPLE 7 *Find the domain and range of the function $f(x) = \frac{1}{2-\sin 3x}$.*

SOLUTION We have, $f(x) = \frac{1}{2-\sin 3x}$

Domain of f: We know that

$$-1 \leq \sin 3x \leq 1 \quad \text{for all } x \in R$$

$$\Rightarrow \quad -1 \leq -\sin 3x \leq 1 \quad \text{for all } x \in R$$

$$\Rightarrow \quad 1 \leq 2-\sin 3x \leq 3 \quad \text{for all } x \in R$$

$\therefore \quad 2 - \sin 3x \neq 0 \quad$ for any $x \in R$

$\Rightarrow \quad f(x) = \dfrac{1}{2 - \sin 3x} \quad$ for all $x \in R$

Hence, domain $(f) = R$

Range of f As discussed above

$1 \leq 2 - \sin 3x \leq 3 \quad$ for all $x \in R$

$\Rightarrow \quad \dfrac{1}{3} \leq \dfrac{1}{2 - \sin 3x} \leq 1 \quad$ for all $x \in R$

$\Rightarrow \quad \dfrac{1}{3} \leq f(x) \leq 1 \quad$ for all $x \in R$

$\Rightarrow \quad f(x) \in \left[\dfrac{1}{3}, 1\right]$

Hence, range $(f) = \left[\dfrac{1}{3}, 1\right]$

EXERCISE 3.3

1. Find the domain of each of the following real valued functions of real variable:

   (i) $f(x) = \dfrac{1}{x}$ (ii) $f(x) = \dfrac{1}{x-7}$ (iii) $f(x) = \dfrac{3x-2}{x+1}$

   (iv) $f(x) = \dfrac{2x+1}{x^2-9}$ (v) $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$ [NCERT]

2. Find the domain of each of the following real valued functions of real variable:

   (i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \dfrac{1}{\sqrt{x^2-1}}$

   (iii) $f(x) = \sqrt{9-x^2}$ (iv) $f(x) = \sqrt{\dfrac{x-2}{3-x}}$

3. Find the domain and range of each of the following real valued functions:

   (i) $f(x) = \dfrac{ax+b}{bx-a}$ (ii) $f(x) = \dfrac{ax-b}{cx-d}$ (iii) $f(x) = \sqrt{x-1}$ [NCERT]

   (iv) $f(x) = \sqrt{x-3}$ (v) $f(x) = \dfrac{x-2}{2-x}$ (vi) $f(x) = |x-1|$ [NCERT]

   (vii) $f(x) = -|x|$ [NCERT] (viii) $f(x) = \sqrt{9-x^2}$ [NCERT]

ANSWERS

1. Domain

   (i) $R - \{0\}$
   (ii) $R - \{7\}$
   (iii) $R - \{-1\}$
   (iv) $R - \{-3, 3\}$
   (v) $R - \{2, 6\}$

2.

	Domain	Range
(i)	$[2, \infty)$	$[0, \infty)$
(ii)	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1] \cup (0, \infty)$
(iii)	$[-3, 3]$	$[0, 3]$
(iv)	$[2, 3)$	$[0, \infty)$

3.

	Domain	Range		Domain	Range
(i)	$R - \left\{\dfrac{a}{b}\right\}$	$R - \left\{\dfrac{a}{b}\right\}$	(ii)	$R - \left\{\dfrac{d}{c}\right\}$	$R - \left\{\dfrac{a}{c}\right\}$
(iii)	$[1, \infty)$	$[0, \infty)$	(iv)	$[3, \infty)$	$[0, \infty)$
(v)	$R - \{2\}$	$\{-1\}$	(vi)	R	$[0, \infty)$
(vii)	R	$(-\infty, 0]$	(viii)	$[-3, 3]$	$[0, 3]$

HINTS TO NCERT & SELECTED PROBLEMS

1. (v) $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12} = \dfrac{(x+1)^2}{(x-6)(x-2)}$ is defined for all x satisfying

   $(x-6)(x-2) \neq 0$ i.e. $x \neq 2, 6$

   $\therefore$ Domain $(f) = R - \{2, 6\}$

3. (iii) $f(x) = \sqrt{x-1}$ is defined for all x satisfying

   $x - 1 \geq 0$ i.e. $x \geq 1$

   So, domain $(f) = [1, \infty)$.

   Let $y = \sqrt{x-1}$. Clearly, $y \geq 0$ for all $x \in [1, \infty)$.

   So, range $(f) = [0, \infty)$.

   (vi) We have, $f(x) = |x-1|$

   Clearly, $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

   Also, $f(x) = |x-1| \geq 0$ for all $x \in R$. So, range $(f) = [0, \infty)$.

   (vii) We have $f(x) = -|x|$

   We observe that $f(x)$ is defined for all $x \in R$. So, domain $(f) = R$.

   Also, $|x| \geq 0$ for all $x \in R \Rightarrow -|x| \leq 0$ for all $x \in R \Rightarrow f(x) \leq 0$ for all $x \in R$.

   So, range $(f) = (-\infty, 0]$.

   (viii) We have,

   $f(x) = \sqrt{9-x^2}$

   Clearly, $f(x)$ takes real values if

   $9 - x^2 \geq 0 \Rightarrow x^2 - 9 \leq 0 \Rightarrow (x-3)(x+3) \leq 0 \Rightarrow x \in [-3, 3]$

   $\therefore$ Domain $(f) = [-3, 3]$

   Also, $f(x) = \sqrt{9-x^2} \geq 0$ for all $x \in [-3, 3]$.

   Let $y = \sqrt{9-x^2}$. Then,

   $y^2 = 9 - x^2 \Rightarrow x^2 + y^2 = 9 \Rightarrow x^2 + y^2 = 9 \Rightarrow x = \sqrt{9-y^2}$

   Clearly, $x \in R$ if $y \in [-3, 3]$. But, $y \geq 0$. Therefore, $y \in [0, 3]$.

   Hence, range $(f) = [0, 3]$

3.10 SOME STANDARD REAL FUNCTIONS AND THEIR GRAPHS

In this section, we shall discuss some standard real functions which frequently occur in the study of calculus.

CONSTANT FUNCTION *If k is a fixed real number, then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.*

Sometimes we also call it the constant function k.

We observe that the domain of the constant function $f(x) = k$ is the set R of all real numbers and range of f is the singleton set $\{k\}$.

The graph of a constant function $f(x) = k$ is a straight line parallel to x-axis (See Fig. 3.8) which is above or below x-axis according as k is positive or negative. If $k = 0$, then the straight line is coincident to x-axis.

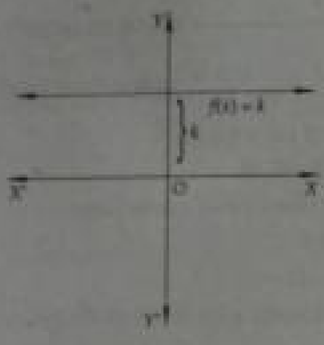


Fig. 3.8

IDENTITY FUNCTION *The function that associates each real number to itself is called the identity function and is usually denoted by I.*

Thus, the function $I: R \to R$ defined by $I(x) = x$ for all $x \in R$ is called the identity function.

Clearly, the domain and range of the identity function are both equal to R.

The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with X-axis.

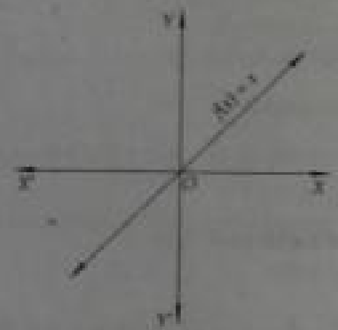


Fig. 3.9

MODULUS FUNCTION *The function $f(x)$ defined by*

$$f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

is called the modulus function.

It is also called the absolute value function.

We observe that the domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers i.e. $R^+ = \{x \in R : x \geq 0\}$.

The graph of the modulus function is as shown in Fig. 3.10. for $x \geq 0$, the graph coincides with the graph of the identity function i.e. the line $y = x$ and for $x < 0$, it is coincident to the line $y = -x$.

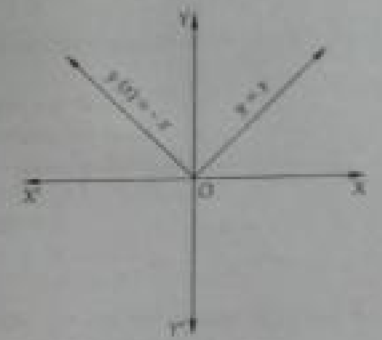


Fig. 3.10

PROPERTIES OF MODULUS FUNCTION

The modulus function has the following properties:

(i) For any real number x, we have

$$\sqrt{x^2} = |x|$$

For example, $\sqrt{\cos^2 x} = |\cos x| = \begin{cases} \cos x, & 0 \leq x \leq \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} < x \leq \pi \end{cases}$

(ii) If a, b are positive real numbers, then

$$x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$$
$$x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a \text{ or, } x \geq a$$
$$x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$$
$$x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a \text{ or, } x > a$$
$$a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$$
$$a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow \in (-b, -a) \cup (a, b)$$

(iii) For real numbers x and y, we have

$$|x+y| = |x| + |y| \Leftrightarrow (x \geq 0 \text{ and } y \geq 0) \text{ or, } (x < 0 \text{ and } y < 0)$$
$$|x-y| = |x| - |y| \Leftrightarrow (x \geq 0 \text{ and } |x| \geq |y|)$$
$$\text{or, } (x \leq 0, y \leq 0 \text{ and } |x| \geq |y|)$$
$$|x \pm y| \leq |x| + |y|$$
$$|x \pm y| \geq \big||x| - |y|\big|$$

GREATEST INTEGER FUNCTION (FLOOR FUNCTION) *For any real number x, we use the symbol [x] or, $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x. For example,*

$$[2.75] = 2, [3] = 3, [0.74] = 0, [-7.45] = -8 \text{ etc.}$$

The function $f: R \to R$ defined by $f(x) = [x]$ for all $x \in R$ is called the greatest integer function or the floor function.

It is also called a step function.

Clearly, domain of the greatest integer function is the set R of all real numbers and the range is the set Z of all integers as it attains only integer values.

The graph of the greatest integer function is shown in Fig 3.11.

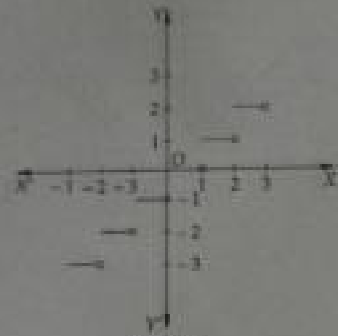


Fig. 3.11

PROPERTIES OF GREATEST INTEGER FUNCTION *If n is an integer and x is a real number between n and $n+1$, then*

(i) $[-n] = -[n]$

(ii) $[x+k] = [x]+k$ for any integer k.

(iii) $[-x] = -[x]-1$

(iv) $[x]+[-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $[x]-[-x] = \begin{cases} 2[x]+1, & \text{if } x \notin Z \\ 2[x], & \text{if } x \in Z \end{cases}$

(vi) $[x] \geq k \Rightarrow x \geq k$, where $k \in Z$

(vii) $[x] \leq k \Rightarrow x < k+1$, where $k \in Z$

(viii) $[x] > k \Rightarrow x \geq k+1$, were $k \in Z$

(ix) $[x] < k \Rightarrow x < k$, where $k \in Z$

(x) $[x+y] = [x] + \big[y + x - [x]\big]$ for all $x, y \in R$

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \dots + \left[x + \frac{n-1}{n}\right] = [nx]$, $n \in N$.

SMALLEST INTEGER FUNCTION (CEILING FUNCTION) *For any real number x, we use the symbol $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.*

For example,

$$\lceil 4.7 \rceil = 5, \lceil -7.2 \rceil = -7, \lceil 5 \rceil = 5, \lceil 0.75 \rceil = 1 \text{ etc.}$$

The function $f: R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ is called the smallest integer function or the ceiling function.

It is also a step function.

We observe that the domain of the smallest integer function is the set R of all real numbers and its range is the set Z of all integers.

The graph of the smallest integer function is as shown in Fig. 3.12.

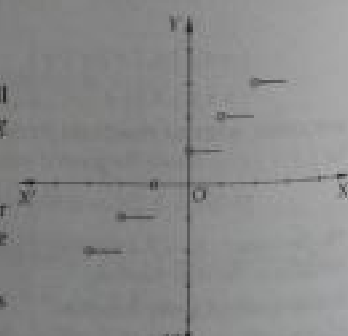


Fig. 3.12

PROPERTIES OF SMALLEST INTEGER FUNCTION Following are some properties of smallest integer function:

(i) $\lceil -n \rceil = -\lceil n \rceil$, where $n \in Z$

(ii) $\lceil -x \rceil = -\lceil x \rceil + 1$, where $x \in R - Z$

(iii) $\lceil x + n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$

(iv) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $\lceil x \rceil - \lceil -x \rceil = \begin{cases} 2\lceil x \rceil - 1, & \text{if } x \notin Z \\ 2\lceil x \rceil, & \text{if } x \in Z \end{cases}$

FRACTIONAL PART FUNCTION *For any real number x, we use the symbol $\{x\}$ to denote the fractional part or decimal part of x. For example,*

$$\{3.45\} = 0.45, \{-2.75\} = 0.25, \{-0.55\} = 0.45, \{3\} = 0, \{-7\} = 0 \text{ etc.}$$

The function $f: R \to R$ defined by $f(x) = \{x\}$ for all $x \in R$ is called the fractional part function.

We observe that the domain of the fractional part function is the set R of all real numbers and the range is the set $[0, 1)$.

It is evident from the definition that

$$f(x) = \{x\} = x - [x] \quad \text{for all } x \in R$$

The graph of the fractional part function is as shown in Fig. 3.13.

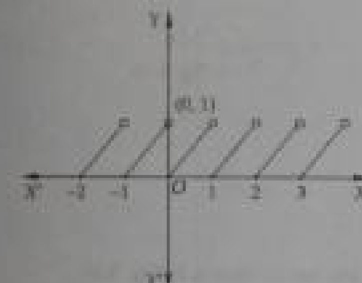


Fig. 3.13

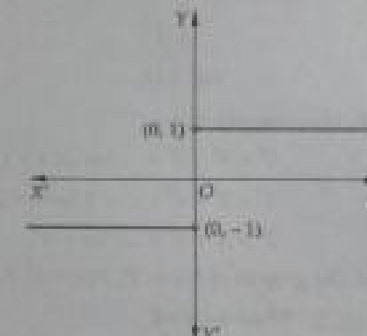


Fig. 3.14

SIGNUM FUNCTION *The function f defined by*

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

or, $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ *is called the signum function.*

The domain of the signum function is the set R of all real numbers and the range is the set $\{-1, 0, 1\}$.

The graph of the signum function is as shown in Fig. 3.14.

EXPONENTIAL FUNCTION *If a is a positive real number other than unity, then a function that associates each $x \in R$ to a^x is called the exponential function.*

In other words, a function $f: R \to R$ defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$ is called the exponential function.

We observe that the domain of an exponential function is R the set of all real numbers and the range is the set $(0, \infty)$ as it attains only positive values.

As $a > 0$ and $a \neq 1$. So, we have the following cases.

CASE I When $a > 1$

We observe that the values of $y = f(x) = a^x$ increase as the values of x increase. Also,

$$f(x) = a^x \begin{cases} < 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ > 1 & \text{for } x > 0. \end{cases}$$

Thus, the graph of $f(x) = a^x$ for $a > 1$ as shown in Fig. 3.15.

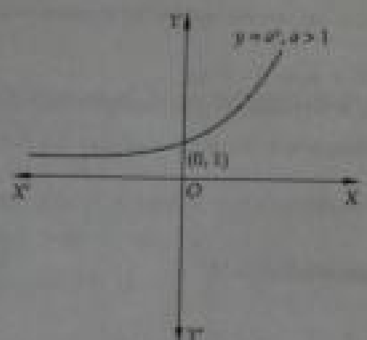


Fig. 3.15

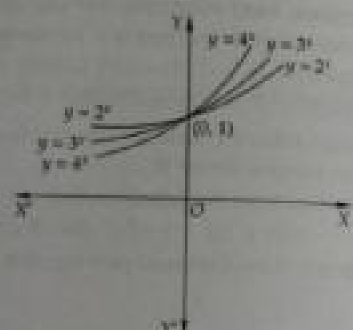


Fig. 3.16

We also observe that

$$2^x < 3^x < 4^x < \ldots \text{ for all } x > 1$$

$$2^x = 3^x = 4^x = \ldots = 1 \text{ for } x = 0$$

$$2^x > 3^x > 4^x > \ldots \text{ for } x < 1$$

So, the graphs of $f(x) = 2^x, f(x) = 3^x, f(x) = 4^x$ etc. are as shown in Fig. 3.16.

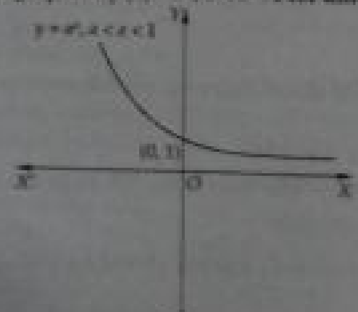
CASE II When $0 < a < 1$

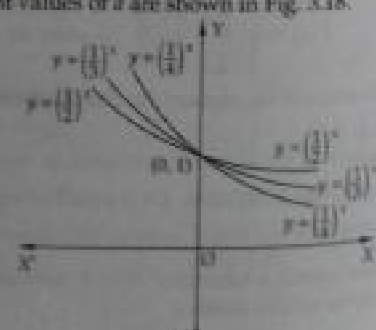
In this case, the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0$ for all $x \in R$.

Also, $$y = f(x) = a^x \begin{cases} > 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ < 1 & \text{for } x > 0 \end{cases}$$

Thus, the graph of $f(x) = a^x$ for $0 < a < 1$ is as shown in Fig. 3.17.

The graphs of $f(x) = a^x$, $0 < a < 1$ for different values of a are shown in Fig. 3.18.





REMARK *We have, $2 < e < 3$. Therefore, graph of $f(x) = e^x$ is identical to that of $f(x) = a^x$ for $a > 1$ and the graph of $f(x) = e^{-x}$ is identical to that of $f(x) = a^x$ for $0 < a < 1$.*

LOGARITHMIC FUNCTION *If $a > 0$ and $a \neq 1$, then the function defined by $f(x) = \log_a x$, $x > 0$ is called the logarithmic function.*

Previously we have learnt that the logarithmic function and the exponential function are inverse functions i.e.

$$\log_a x = y \Leftrightarrow x = a^y$$

We observe that the domain of the logarithmic function is the set of all non-negative real numbers i.e. $(0, \infty)$ and the range is the set R of all real numbers.

As $a > 0$ and $a \neq 1$. So, we have the following cases.

CASE I When $a > 1$

In this case, we have

$$y = \log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1 \end{cases}$$

Also, the values of y increase with the increase in x.

So, the graph of $y = \log_a x$ is as shown in Fig. 3.19.

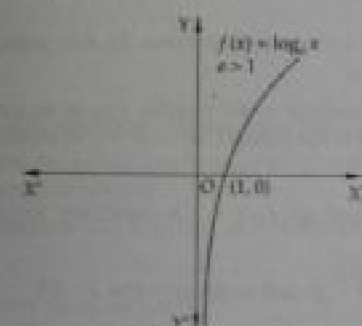


Fig. 3.19

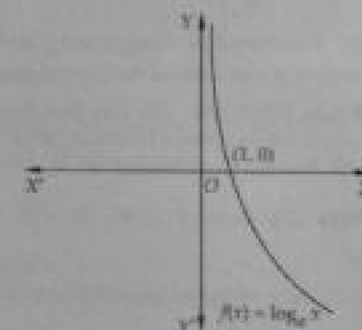


Fig. 3.20

CASE II When $0 < a < 1$

In this case, we have

$$y = \log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$$

Also, the values of y decrease with the increase in x.

So, the graph of $y = \log_a x$ is as shown in Fig. 3.20.

PROPERTIES OF LOGARITHMIC FUNCTION Following are some useful properties of logarithmic function:

(i) $\log_a 1 = 0$, where $a > 0, a \neq 1$

(ii) $\log_a a = 1$, where $a > 0, a \neq 1$

(iii) $\log_a (xy) = \log_a |x| + \log_a |y|$, where $a > 0, a \neq 1$ and $xy > 0$

(iv) $\log_a\left(\frac{x}{y}\right) = \log_a |x| - \log_a |y|$, where $a > 0, a \neq 1$ and $\frac{x}{y} > 0$

(v) $\log_a (x^n) = n \log_a |x|$, where $a > 0, a \neq 1$ and $x^n > 0$

(vi) $\log_{a^n} x^m = \frac{m}{n} \log_{|a|} |x|$, where $a > 0, a \neq 1$ and $x > 0$

(vii) $x^{\log_a y} = y^{\log_a x}$, where $x > 0, y > 0, a > 0, a \neq 1$

(viii) If $a > 1$, then the values of $f(x) = \log_a x$ increase with the increase in x i.e.

$$x < y \Leftrightarrow \log_a x < \log_a y$$

Also,
$$\log_a x \begin{cases} < 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ > 0 & \text{for } x > 1. \end{cases}$$

(ix) If $0 < a < 1$, then the values of $f(x) = \log_a x$ decrease with the increase in x i.e.

$$x < y \Leftrightarrow \log_a x > \log_a y$$

Also,
$$\log_a x \begin{cases} > 0 & \text{for } 0 < x < 1 \\ = 0 & \text{for } x = 1 \\ < 0 & \text{for } x > 1 \end{cases}$$

(x) $\log_a x = \frac{1}{\log_x a}$ for $a > 0, a \neq 1$ and $x > 0, x \neq 1$.

REMARK *Functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other. So, their graphs are mirror images of each other in the line mirror $y = x$.*

RECIPROCAL FUNCTION The function that associates a real number x to its reciprocal $1/x$ is called the reciprocal function. Since $1/x$ is not defined for $x = 0$. So, we define the reciprocal function as follows:

DEFINITION *The function $f: R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called the reciprocal function.*

Clearly, domain of the reciprocal function is $R - \{0\}$ and its range is also $R - \{0\}$.

We observe that the sign of $\frac{1}{x}$ is same as that of x and $\frac{1}{x}$ decreases with the increase in x. So, the graph of $f(x) = \frac{1}{x}$ is as shown in Fig. 3.21.

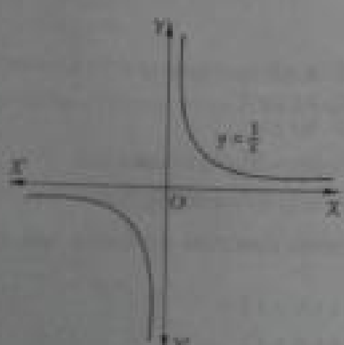


Fig. 3.21

SQUARE ROOT FUNCTION The function that associates a real number x to $+\sqrt{x}$ is called the square root function. Since $\sqrt{x}$ is real for $x \geq 0$. So, we defined the square root function as follows:

DEFINITION *The function $f: R^+ \to R$ defined by*

$$f(x) = +\sqrt{x}$$

is called the square root function.

Clearly, domain of the square root function is R^+ i.e. $[0, \infty)$ and its range is also $[0, \infty)$.

We observe that the values of $f(x) = +\sqrt{x}$ increase with the increase in x. So, the graph of $f(x) = +\sqrt{x}$ is as shown in Fig. 3.22.

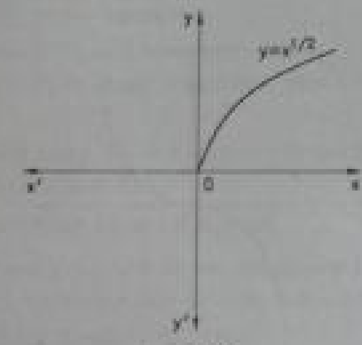


Fig. 3.22

SQUARE FUNCTION The function that associates a real number x to its square i.e. x^2 is called the square function. Since x^2 is defined for all $x \in R$. So, we define the square function as follows:

DEFINITION *The function $f: R \to R$ defined by*

$$f(x) = x^2$$

is called the square function.

Clearly, domain of the square function is R and its range is the set of all non-negative real numbers i.e. $[0, \infty)$. The graph of $f(x) = x^2$ is parabola as shown in Fig. 3.23.

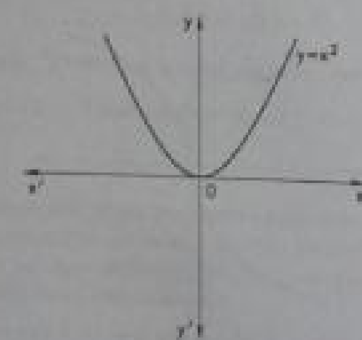


Fig. 3.23

CUBE FUNCTION The function that associate a real number x to its cube is called the cube function. We observe that x^3 is meaningful for all $x \in R$. So, we define the cube function as follows:

DEFINITION *The function $f : R \to R$ defined by*

$$f(x) = x^3$$

is called the cube function.

We observe that the sign of x^3 is same as that of x and the values of x^3 increase with the increase in x. So, the graph of $f(x) = x^3$ is as shown in Fig. 3.24. Clearly, the graph is symmetrical in opposite quadrants.

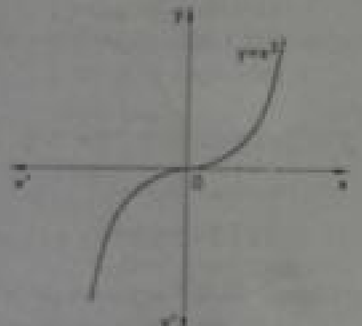


Fig. 3.24

CUBE ROOT FUNCTION The function that associates a real number x to its cube root $x^{1/3}$ is called the cube root function. Clearly, $x^{1/3}$ is defined for all $x \in R$. So, we define the cube root function as follows:

DEFINITION *The function $f : R \to R$ defined by $f(x) = x^{1/3}$ is called the cube root function.*

Clearly, domain and range of the cube root function are both equal to R.

Also, the sign of $x^{1/3}$ is same as that of x and $x^{1/3}$ increase with the increase in x. So, the graph of $f(x)\, x^{1/3}$ is as shown in Fig. 3.25.

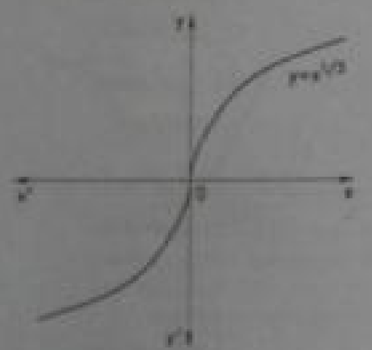


Fig. 3.25

REMARK 1 *A function $f : R \to R$ is said to be a polynomial function if $f(x)$ is a polynomial in x. For example, $f(x) = x^2 - x + 4$, $g(x) = x^3 + 3x^2 + \sqrt{2}\,x - 1$ etc are polynomial functions.*

REMARK 2 *A function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. The domain of a rational function $f(x) = \frac{p(x)}{q(x)}$ is the set of all real numbers, except points where $q(x) = 0$.*

3.11 OPERATIONS ON REAL FUNCTIONS

In this section, we shall introduce various operations, namely addition, subtraction, multiplication, division etc. on real functions.

ADDITION *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their sum $f + g$ is defined as that function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number $f(x) + g(x)$.*

In other words, if $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions, then their sum $f + g$ is a function from $D_1 \cap D_2$ to R such that

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in D_1 \cap D_2$$

PRODUCT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their product (or pointwise multiplication) fg is a function from $D_1 \cap D_2$ to R and is defined as*

$$(fg)(x) = f(x)\,g(x) \quad \text{for all } x \in D_1 \cap D_2$$

DIFFERENCE (SUBTRACTION) *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the difference of g from f is denoted by $f - g$ and is defined as*

$$(f - g)(x) = f(x) - g(x) \quad \text{for all } x \in D_1 \cap D_2$$

QUOTIENT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the quotient of f by g is denoted by $\frac{f}{g}$ and it is a function from $D_1 \cap D_2 - \{x : g(x) = 0\}$ to R defined by*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{for all } x \in D_1 \cap D_2 - \{x : g(x) = 0\}$$

MULTIPLICATION OF A FUNCTION BY A SCALAR *Let $f : D \to R$ be a real function and α be a scalar (real number). Then the product αf is a function from D to R and is defined as*

$$(\alpha f)(x) = \alpha f(x) \quad \text{for all } x \in D.$$

RECIPROCAL OF A FUNCTION *If $f : D \to R$ is a real function, then its reciprocal function $\frac{1}{f}$ is a function from $D - \{x : f(x) = 0\}$ to R and is defined as*

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$$

REMARK 1 *The sum, difference product and quotient are defined for real functions only on their common domain. These operations do not make any sense for general functions even if their domains are same, because the sum, difference, product and quotient may or may not be meaningful for the elements in their common domain.*

REMARK 2 *For any real function $f : D \to R$ and $n \in N$, we define*

$$(\underbrace{fff \dots f}_{n\text{-times}})(x) = \underbrace{f(x) f(x) \dots f(x)}_{n\text{-times}} = \{f(x)\}^n \quad \text{for all } x \in D$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Find the sum and difference of the identity function and the modulus function.* [NCERT]

SOLUTION We know that $f : R \to R$ defined by $f(x) = x$ is the identity function and $g : R \to R$ defined by $g(x) = |x|$ is the modulus function.

Clearly, f and g have the same domain. Therefore,

$$f + g : R \to R \text{ and } f - g : R \to R$$

Now,

$$(f+g)(x) = f(x) + g(x)$$

$$\Rightarrow \quad (f+g)(x) = x + |x|$$

$$\Rightarrow \quad (f+g)(x) = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad (f+g)(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and,

$$(f-g)(x) = f(x) - g(x)$$

$$\Rightarrow \quad (f-g)(x) = x - |x|$$

$$\Rightarrow \quad (f-g)(x) = \begin{cases} x - x, & \text{if } x \geq 0 \\ x - (-x), & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad (f-g)(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

Thus, $f + g : R \to R$ and $f - g : R \to R$ are defined as

$$(f+g)(x) = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

and, $$(f-g)(x) = \begin{cases} 0, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

EXAMPLE 2 *What are the sum and difference of the identity function and the reciprocal function?* [NCERT]

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then, $f : R \to R$ and $g : R - \{0\} \to R$ such that

$$f(x) = x \quad \text{for all } x \in R$$

and, $$g(x) = \frac{1}{x} \quad \text{for all } x \in R - \{0\}$$

The domains of f and g are R and $R - \{0\}$ respectively.

Also, we have

$$R \cap R - \{0\} = R - \{0\}.$$

Therefore, $f + g : R - \{0\} \to R$ and $f - g : R - \{0\} \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = x + \frac{1}{x}$$

and, $$(f-g)(x) = f(x) - g(x) = x - \frac{1}{x}$$

EXAMPLE 3 *Let $f : [2, \infty) \to R$ and $g : [-2, \infty) \to R$ be two real functions defined by*

$$f(x) = \sqrt{x-2} \text{ and } g(x) = \sqrt{x+2}.$$

Find $f + g$ and $f - g$.

SOLUTION Here, we have

$$D_1 = [2, \infty) \text{ and } D_2 = [-2, \infty)$$

$$\therefore \quad D_1 \cap D_2 = [2, \infty).$$

Thus, $f + g : [2, \infty) \to R$ and $f - g : [2, \infty) \to R$ are given by

$$(f+g)(x) = f(x) + g(x) = \sqrt{x-2} + \sqrt{x+2} \quad \text{for all } x \in [2, \infty)$$

and, $$(f-g)(x) = f(x) - g(x) = \sqrt{x-2} - \sqrt{x+2} \quad \text{for all } x \in [2, \infty).$$

EXAMPLE 4 *Find the product of the identity function and the modulus function.*

SOLUTION Let f and g denote respectively the identity function and modulus function. Then,

$$f : R \to R \text{ such that } f(x) = x \quad \text{for all } x$$

and,

$$g : R \to R \text{ such that } g(x) = |x| \quad \text{for all } x.$$

Clearly, f and g have the same domain. Therefore, the product fg is a function from R to itself and is given by

$$(fg)(x) = f(x)\, g(x)$$

$$\Rightarrow \quad (fg)(x) = x\,|x|$$

$$\Rightarrow \quad (fg)(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

EXAMPLE 5 *Find the quotient of the identity function by the modulus function.*

SOLUTION Let f and g denote respectively the identity function and the modulus function. Then,

$$f : R \to R \text{ is defined as } f(x) = x \text{ and,}$$

$$g : R \to R \text{ is defined as } g(x) = |x|.$$

Clearly, f and g have the same domain.

Also, $g(x) = 0 \Rightarrow |x| = 0 \Rightarrow x = 0.$

Therefore, the quotient of f by g i.e. $\frac{f}{g}$ is a function from $R - \{0\} \to R$ and is defined as

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} \frac{x}{x} = 1, & x > 0 \\ \frac{x}{-x} = -1, & x < 0 \end{cases}$$

EXAMPLE 6 *Find the product of the identity function and the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then,

$$f : R \to R \text{ is defined as } f(x) = x \text{ for all } x \in R \text{ and,}$$

$$g : R - \{0\} \to R \text{ is defined as } g(x) = \frac{1}{x} \text{ for all } x \in R - \{0\}$$

We have,

$$\text{Domain}(f) \cap \text{Domain}(g) = R \cap R - \{0\} = R - \{0\}$$

Therefore, the product fg is a function from $R-\{0\}$ to R and is defined as

$$(fg)(x) = f(x)\,g(x) = x \times \frac{1}{x} = 1 \text{ for all } x \in R-\{0\}$$

Thus, $fg : R-\{0\} \to R$ is given by

$$(fg)(x) = 1 \text{ for all } x \in R-\{0\}$$

EXAMPLE 7 *Find the quotient of the identity function by the reciprocal function.*

SOLUTION Let f and g denote respectively the identity function and the reciprocal function. Then,

$f : R \to R$ is defined as $f(x) = x$ for all $x \in R$ and,

$g : R-\{0\} \to R$ is defined as $g(x) = \frac{1}{x}$ for all $x \in R-\{0\}$

We have,

$$\text{Domain}(f) \cap \text{Domain}(g) = R \cap R-\{0\} = R-\{0\}$$

Also, $g(x) \neq 0$ for any $x \in R-\{0\}$

$\therefore$ $\frac{f}{g}$ is a function from $R-\{0\} \to R$ and is given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{1/x} = x^2$$

Hence, $\frac{f}{g} : R-\{0\} \to R$ is given by

$$\left(\frac{f}{g}\right)(x) = x^2 \text{ for all } x \in R-\{0\}.$$

EXAMPLE 8 *Let c be a non-zero real number and $f : R \to R$ be a function defined by $f(x) = \frac{x}{c}$ for all $x \in R$. Find (i) cf (ii) $c^2 f$ (iii) $\left(\frac{1}{c}\right)f$.*

SOLUTION Clearly, cf, $c^2 f$ and $\left(\frac{1}{c}\right)f$ are functions from R to itself.

Now,

(i) $(cf)(x) = c\,f(x) = c \times \frac{x}{c} = x$ for all $x \in R$

(ii) $(c^2 f)(x) = c^2 f(x) = c^2 \times \frac{x}{c} = cx$ for all $x \in R$

(iii) $\left(\left(\frac{1}{c}\right)f\right)(x) = \left(\frac{1}{c}\right)f(x) = \frac{1}{c} \times \frac{x}{c} = \frac{x}{c^2}$ for all $x \in R$.

EXAMPLE 9 *Let f and g be two real functions defined by*

$$f(x) = \frac{1}{x+4} \text{ and } g(x) = (x+4)^3$$

Find the following:

(i) $f+g$ (ii) $f-g$ (iii) fg (iv) $\frac{f}{g}$

(v) $2f$ (vi) $\frac{1}{f}$ (vii) $\frac{1}{g}$

SOLUTION We observe that $f(x) = \frac{1}{x+4}$ is defined for all $x \neq -4$.

So, $\text{domain}(f) = R-\{-4\}$.

Clearly, $g(x) = (x+4)^3$ is defined for all $x \in R$.

So, $\text{domain}(g) = R$.

$\therefore$ $\text{Domain}(f) \cap \text{Domain}(g) = R-\{-4\}$.

(i) Clearly, $f+g : R-\{-4\} \to R$ is given by

$(f+g)(x) = f(x) + g(x)$

$\Rightarrow$ $(f+g)(x) = \frac{1}{x+4} + (x+4)^3 = \frac{(x+4)^4+1}{x+4}$

(ii) We find that $f-g : R-\{-4\} \to R$ is defined as

$(f-g)(x) = f(x) - g(x)$

$\Rightarrow$ $(f-g)(x) = \frac{1}{x+4} - (x+4)^3 = \frac{1-(x+4)^4}{x+4}$

(iii) $fg : R-\{-4\} \to R$ is given by

$(fg)(x) = f(x) \cdot g(x)$

$\Rightarrow$ $(fg)(x) = \frac{1}{x+4} \times (x+4)^3 = (x+4)^2$

(iv) We have, $g(x) = (x+4)^3$

$\therefore$ $g(x) = 0 \Rightarrow (x+4)^3 = 0 \Rightarrow x = -4$.

So, $\text{Domain}\left(\frac{f}{g}\right) = \text{Domain}(f) \cap \text{Domain}(g) - \{x : g(x) = 0\} = R-\{-4\}$

Thus, $\frac{f}{g} : R-\{-4\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{(x+4)^4}$

(v) We have,

$$(2f)(x) = 2\big(f(x)\big) = \frac{2}{x+4} \text{ for all } x \in R-\{-4\}.$$

(vi) We observe that $f(x) \neq 0$ for any $x \in R-\{-4\}$

$\therefore$ $\frac{1}{f} : R-\{-4\} \to R$ is given by

$$\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{1/(x+4)} = (x+4)$$

(vii) We observe that $g(x) = (x+4)^3 = 0$ for $x = -4$.

$\therefore$ $\frac{1}{g} : R-\{-4\} \to R$ is given by

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} = \frac{1}{(x+4)^3}$$

EXAMPLE 10 *Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Then, find each of the following functions:*

(i) $f+g$ (ii) $f-g$ (iii) fg

(iv) $\frac{f}{g}$ (v) ff (vi) gg

SOLUTION We have,

$f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$

Clearly, $f(x)$ is defined for

$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$

$\therefore$ Domain $(f) = [-2, \infty)$

$g(x)$ is defined for

$4 - x^2 \geq 0 \Rightarrow x^2 - 4 \leq 0 \Rightarrow (x-2)(x+2) \leq 0 \Rightarrow x \in [-2, 2]$

$\therefore$ Domain $(g) = [-2, 2]$

Now,

Domain $(f) \cap$ Domain $(g) = [-2, \infty) \cap [-2, 2] = [-2, 2]$

(i) $f + g : [-2, 2] \rightarrow R$ is given by

$(f+g)(x) = f(x) + g(x) = \sqrt{x+2} + \sqrt{4-x^2}$

(ii) $f - g : [-2, 2] \rightarrow R$ is given by

$(f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \sqrt{4-x^2}$.

(iii) $fg : [-2, 2] \rightarrow R$ is given by

$(fg)(x) = f(x)\,g(x)$

$\Rightarrow$ $(fg)(x) = \sqrt{x+2} \times \sqrt{4-x^2}$

$\Rightarrow$ $(fg)(x) = \sqrt{(x+2)^2(2-x)} = (x+2)\sqrt{2-x}$

(iv) We have,

$g(x) = \sqrt{4-x^2}$

$\therefore$ $g(x) = 0 \Rightarrow 4 - x^2 = 0 \Rightarrow x = \pm 2$

So, Domain $\left(\frac{f}{g}\right) = [-2, 2] - \{-2, 2\} = (-2, 2)$

$\therefore$ $\frac{f}{g} : (-2, 2) \rightarrow R$ is given by

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{(x+2)}}{\sqrt{4-x^2}} = \frac{1}{\sqrt{2-x}}$

(v) We have,

$(ff)(x) = f(x) f(x) = \{f(x)\}^2 = (\sqrt{x+2})^2 = x + 2$ for all $x \in [-2, \infty)$

(vi) We have,

$(gg)(x) = g(x)g(x) = \{g(x)\}^2 = \left(\sqrt{4-x^2}\right)^2 = 4 - x^2$ for all $x \in [-2, 2]$

EXAMPLE 11 *Let f be the exponential function and g be the logarithmic function. Find:*

(i) $(f+g)(1)$ (ii) $(fg)(1)$ (iii) $(3f)(1)$ (iv) $(5g)(1)$

SOLUTION We have,

$f : R \rightarrow R$ given by $f(x) = e^x$

and, $g : R^+ \rightarrow R$ given by $g(x) = \log_e x$.

(i) Since Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore,

$f + g : R^+ \rightarrow R$ is given by

$(f+g)(x) = f(x) + g(x) = e^x + \log_e x$ for all $x \in R^+$

Clearly, $1 \in R^+$

$\therefore$ $(f+g)(1) = f(1) + g(1) = e^1 + \log_e 1 = e + 0 = e$.

(ii) We have,

Domain $(f) \cap$ Domain $(g) = R \cap R^+ = R^+$. Therefore,

$fg : R^+ \rightarrow R$ is given by

$(fg)(x) = f(x)\,g(x) = e^x \cdot \log_e x$.

$\Rightarrow$ $(fg)(1) = f(1)\,g(1) = e^1 \times \log_e 1 = e \times 0 = 0$

(iii) We have,

$(3f)(x) = 3(f(x)) = 3e^x$

$\therefore$ $(3f)(1) = 3e^1 = 3e$

(iv) We have,

$(5g)(x) = 5(g(x)) = 5\log_e x$

$\therefore$ $(5g)(1) = 5\log_e 1 = 5 \times 0 = 0$.

EXAMPLE 12 *Find the domain of each of the following function given by*

(i) $f(x) = \frac{1}{\sqrt{x - |x|}}$ (ii) $f(x) = \frac{1}{\sqrt{x + |x|}}$

(iii) $f(x) = \frac{1}{\sqrt{x - [x]}}$ (vi) $f(x) = \frac{1}{\sqrt{x + [x]}}$

SOLUTION We have,

(i) $f(x) = \frac{1}{\sqrt{x - |x|}}$

We know that,

$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$\Rightarrow$ $x - |x| = \begin{cases} x - x = 0, & \text{if } x \geq 0 \\ x + x = 2x, & \text{if } x < 0 \end{cases}$

$\Rightarrow$ $x - |x| \leq 0$ for all x

$\Rightarrow$ $\frac{1}{\sqrt{x - |x|}}$ does not take real values for any $x \in R$

$\Rightarrow$ $f(x)$ is not defined for any $x \in R$.

Hence, Domain $(f) = \phi$.

(ii) $f(x) = \frac{1}{\sqrt{x + |x|}}$

We know that,

$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

$\Rightarrow$ $x + |x| = \begin{cases} x + x, & \text{if } x \geq 0 \\ x - x, & \text{if } x < 0 \end{cases}$

$\Rightarrow$ $x + |x| = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$ ...(i)

Now, $f(x) = \frac{1}{\sqrt{x + |x|}}$ assumes real values, if

$x + |x| > 0$

$\Rightarrow \quad x > 0$ [Using (i)]

$\Rightarrow \quad x \in (0, \infty).$

Hence, Domain $(f) = (0, \infty)$.

(iii) $f(x) = \dfrac{1}{\sqrt{x - [x]}}$

We know that $0 \le x - [x] < 1$ for all $x \in R$. Also, $x - [x] = 0$ for $x \in Z$.

Now, $f(x) = \dfrac{1}{\sqrt{x - [x]}}$ is defined if

$x - [x] > 0$

$\Rightarrow \quad x \in R - Z$ $\left[\because x - [x] = 0 \text{ for } x \in Z \text{ and } 0 < x - [x] < 1 \text{ for } x \in R - Z\right]$

Hence, Domain $(f) = R - Z$.

(iv) $f(x) = \dfrac{1}{\sqrt{x + [x]}}$

We know that

$x + [x] > 0$ for all $x > 0$

$x + [x] = 0$ for $x = 0$

$x + [x] < 0$ for all $x < 0$. ...(i)

Also, $f(x) = \dfrac{1}{\sqrt{x + [x]}}$ is defined for all x satisfying $x + [x] > 0$

Hence, Domain $(f) = (0, \infty)$ [Using (i)]

EXAMPLE 13 *Find the domain of definition of the function $f(x)$ given by*

$$f(x) = \log_4 \left\{\log_5 \left(\log_3 (18x - x^2 - 77)\right)\right\}$$

SOLUTION We have,

$$f(x) = \log_4 \left\{\log_5 \left(\log_3 (18x - x^2 - 77)\right)\right\}$$

Since $\log_a x$ is defined for all $x > 0$. Therefore, $f(x)$ is defined if

$\log_5 \left\{\log_3 (18x - x^2 - 77)\right\} > 0$ and $18x - x^2 - 77 > 0$

$\Rightarrow \quad \log_3 (18x - x^2 - 77) > 5^0$ and $x^2 - 18x + 77 < 0$

$\Rightarrow \quad \log_3 (18x - x^2 - 77) > 1$ and $(x - 11)(x - 7) < 0$

$\Rightarrow \quad 18x - x^2 - 77 > 3^1$ and $7 < x < 11$

$\Rightarrow \quad 18x - x^2 - 80 > 0$ and $7 < x < 11$

$\Rightarrow \quad x^2 - 18x + 80 < 0$ and $7 < x < 11$

$\Rightarrow \quad (x - 10)(x - 8) < 0$ and $7 < x < 11$

$\Rightarrow \quad 8 < x < 10$ and $7 < x < 11$

$\Rightarrow \quad 8 < x < 10$

$\Rightarrow \quad x \in (8, 10).$

Hence, the domain of $f(x)$ is $(8, 10)$.

EXAMPLE 14 *Find the domain of definition of the function $f(x)$ given by*

$$f(x) = \frac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$$

SOLUTION We have,

$$f(x) = \frac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$$

Let, $g(x) = \dfrac{1}{\log_{10}(1 - x)}$ and $h(x) = \sqrt{x + 2}$. Then,

$f(x) = g(x) + h(x)$

$\Rightarrow \quad$ Domain (f) = Domain $(g) \cap$ Domain (h)

Now, $g(x) = \dfrac{1}{\log_{10}(1 - x)}$ is defined, if

$\log_{10}(1 - x)$ is defined and $\log_{10}(1 - x) \ne 0$

$\Rightarrow \quad 1 - x > 0$ and $1 - x \ne 1$

$\Rightarrow \quad x < 1$ and $x \ne 0$

$\Rightarrow \quad x \in (-\infty, 0) \cup (0, 1)$

$\therefore \quad$ Domain $(g) = (-\infty, 0) \cup (0, 1)$.

$h(x) = \sqrt{x + 2}$ is defined, if

$x + 2 \ge 0 \Rightarrow x \ge -2 \Rightarrow x \in [-2, \infty)$.

$\therefore \quad$ Domain $(h) = [-2, \infty)$.

Hence, Domain $(f) = (-\infty, 0) \cup (0, 1) \cap [-2, \infty)$

$= [-2, 0) \cup (0, 1)$

EXAMPLE 15 *Find the range of each of the following functions:*

(i) $f(x) = |x - 3|$ (ii) $f(x) = 1 - |x - 2|$ (iii) $f(x) = \dfrac{|x - 4|}{x - 4}$

SOLUTION

(i) We have,

$f(x) = |x - 3|$

Clearly, $f(x)$ is defined for all $x \in R$

$\therefore \quad$ Domain $(f) = R$.

Now,

$|x - 3| \ge 0$ for all $x \in R$

$\Rightarrow \quad 0 \le |x - 3| < \infty$ for all $x \in R$

$\Rightarrow \quad 0 \le f(x) < \infty$ for all $x \in R$

$\Rightarrow \quad f(x) \in [0, \infty)$ for all $x \in R$

Hence, Range $(f) = [0, \infty)$.

(ii) We have,

$f(x) = 1 - |x - 2|$.

We observe that $f(x)$ is defined for all $x \in R$.

$\therefore \quad$ Domain $(f) = R$.

Now,

$0 \le |x - 2| < \infty$ for all $x \in R$

$\Rightarrow \quad -\infty < -|x - 2| \le 0$ for all $x \in R$

$\Rightarrow \quad -\infty < 1 - |x - 2| \le 1$ for all $x \in R$

$\Rightarrow \quad -\infty < f(x) \le 1$ for all $x \in R$

$\Rightarrow \quad f(x) \in (-\infty, 1]$

Hence, Range $(f) = (-\infty, 1]$

(iii) We have,

$$f(x) = \frac{|x-4|}{x-4}$$

Clearly, $f(x)$ is defined for all $x \in R$ except at $x = 4$.

$\therefore$ Domain $(f) = R - \{4\}$

Now,

$$f(x) = \frac{|x-4|}{x-4}$$

$$f(x) = \begin{cases} \frac{x-4}{x-4} = 1, & \text{if } x > 4 \\ \frac{-(x-4)}{x-4} = -1, & \text{if } x < 4 \end{cases}$$

$\therefore$ Range $(f) = \{-1, 1\}$

EXAMPLE 16 *Find the domain and range of each of the following functions given by*

(i) $f(x) = \frac{1}{\sqrt{x - [x]}}$ (ii) $f(x) = 1 - |x - 3|$

SOLUTION

(i) We have,

$$f(x) = \frac{1}{\sqrt{x - [x]}}$$

Domain of f: We know that

$0 \le x - [x] < 1$ for all $x \in R$

Also, $x - [x] = 0$ for $x \in Z$.

$\therefore$ $0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow$ $f(x) = \frac{1}{\sqrt{x-[x]}}$ exists for all $x \in R - Z$

$\Rightarrow$ Domain $(f) = R - Z$.

Range of f: We have,

$0 < x - [x] < 1$ for all $x \in R - Z$

$\Rightarrow$ $0 < \sqrt{x - [x]} < 1$ for all $x \in R - Z$

$\Rightarrow$ $1 < \frac{1}{\sqrt{x-[x]}} < \infty$ for all $x \in R - Z$

$\Rightarrow$ $1 < f(x) < \infty$ for all $x \in R - Z$

$\Rightarrow$ Range $(f) = (1, \infty)$.

(ii) We have,

$$f(x) = 1 - |x - 3|$$

Clearly, $f(x)$ is defined for all $x \in R$.

$\therefore$ Domain $(f) = R$.

Range of f: For $x \in R$, we have

$|x - 3| \ge 0$

$\Rightarrow$ $-|x - 3| \le 0$

$\Rightarrow$ $1 - |x - 3| \le 1$

$\Rightarrow$ $f(x) \le 1$

$\Rightarrow$ $f(x) \in (-\infty, 1]$

Hence, Range $(f) = (-\infty, 1]$.

EXAMPLE 17 *Find the domain of the real function $f(x)$ defined by*

$$f(x) = \sqrt{\frac{1-|x|}{2-|x|}}$$

SOLUTION We have,

$$f(x) = \sqrt{\frac{1-|x|}{2-|x|}}$$

We observe that $f(x)$ is defined for all x satisfying

$\frac{1-|x|}{2-|x|} \ge 0$

$\Rightarrow$ $\frac{|x|-1}{|x|-2} \ge 0$

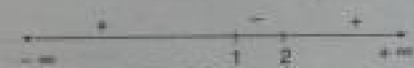


Fig. 3.28

$\Rightarrow$ $|x| \le 1$ or, $|x| > 2$

$\Rightarrow$ $x \in [-1, 1]$ or, $x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow$ $x \in (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$

Hence, domain $(f) = (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$.

EXERCISE 3.4

1. Find $f + g$, $f - g$, cf $(c \in R, c \ne 0)$, fg, $\frac{1}{f}$ and $\frac{f}{g}$ in each of the following:
   (i) If $f(x) = x^3 + 1$ and $g(x) = x + 1$
   (ii) If $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{x+1}$.
2. Let $f(x) = 2x + 5$ and $g(x) = x^2 + x$. Describe (i) $f + g$ (ii) $f - g$ (iii) fg (iv) f/g. Find the domain in each case.
3. If $f(x)$ be defined on $[-2, 2]$ and is given by $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$. Find $g(x)$.
4. Let f, g be two real functions defined by $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{9 - x^2}$. Then, describe each of the following functions:
   (i) $f + g$ (ii) $g - f$ (iii) fg
   (iv) $\frac{f}{g}$ (v) $\frac{g}{f}$ (vi) $2f - \sqrt{5}\,g$
   (vii) $f^2 + 7f$ (viii) $\frac{5}{g}$
5. If $f(x) = \log_e (1 - x)$ and $g(x) = [x]$, then determine each of the following functions:
   (i) $f + g$ (ii) fg (iii) $\frac{f}{g}$ (iv) $\frac{g}{f}$
   Also, find $(f + g)(-1)$, $(fg)(0)$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$, $\left(\frac{g}{f}\right)\left(\frac{1}{2}\right)$.
6. If f, g, h are real functions defined by $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$ and $h(x) = 2x^2 - 3$, then find the values of $(2f + g - h)(1)$ and $(2f + g - h)(0)$.

7. The function f is defined by $f(x)=\begin{cases}1-x, & x<0\\ 1, & x=0\\ x+1, & x>0\end{cases}$. Draw the graph of $f(x)$. [NCERT]

8. Let $f, g: R\to R$ be defined, respectively by $f(x)=x+1$, $g(x)=2x-3$. Find $f+g$, $f-g$ and $\frac{f}{g}$. [NCERT]

9. Let $f:[0,\infty)\to R$ and $g:R\to R$ be defined by $f(x)=\sqrt{x}$ and $g(x)=x$. Find $f+g$, $f-g$, fg and $\frac{f}{g}$. [NCERT]

10. Let $f(x)=x^2$ and $g(x)=2x+1$ be two real functions. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$ and $\left(\frac{f}{g}\right)(x)$. [NCERT]

ANSWERS

1. (i) $f+g:R\to R$ given by $(f+g)(x)=x^3+x+2$
$f-g:R\to R$ given by $(f-g)(x)=x^3-x$
$cf:R\to R$ given by $(cf)(x)=c(x^3+1)$
$fg:R\to R$ given by $(fg)(x)=(x+1)^2(x^2-x+1)$
$\frac{1}{f}:R-\{-1\}\to R$ given by $\left(\frac{1}{f}\right)(x)=\frac{1}{x^3+1}$
$\frac{f}{g}:R-\{-1\}\to R$ given by $\left(\frac{f}{g}\right)(x)=x^2-x+1$
(ii) $f\pm g:[1,\infty)\to R$ defined by $(f\pm g)(x)=\sqrt{x-1}\pm\sqrt{x+1}$
$cf:[1,\infty)\to R$ defined by $(cf)(x)=c\sqrt{x-1}$
$fg:[1,\infty)\to R$ defined by $(fg)(x)=\sqrt{x^2-1}$
$\frac{1}{f}:(1,\infty)\to R$ defined by $\left(\frac{1}{f}\right)(x)=\frac{1}{\sqrt{x-1}}$
$\frac{f}{g}:[1,\infty)\to R$ defined by $\left(\frac{f}{g}\right)(x)=\sqrt{\frac{x-1}{x+1}}$

2. (i) $(f+g)(x)=x^2+3x+5$; dom $(f+g)=R$
(ii) $(f-g)(x)=5+x-x^2$; dom $(f-g)=R$
(iii) $(fg)(x)=2x^3+7x^2+5x$; dom $(fg)=R$
(iv) $\left(\frac{f}{g}\right)(x)=\frac{2x+5}{x^2+x}$, dom $\left(\frac{f}{g}\right)=R-\{0,1\}$

3. $g(x)=\begin{cases}-x, & -2\le x\le 0\\ 0, & 0<x<1\\ 2(x-1), & 1\le x\le 2\end{cases}$

4. (i) $f+g:[-1,3]\to R$ defined by $(f+g)(x)=\sqrt{x+1}+\sqrt{9-x^2}$
(ii) $g-f:[-1,3]\to R$ defined by $(g-f)(x)=\sqrt{9-x^2}-\sqrt{x+1}$
(iii) $fg:[-1,3]\to R$ defined by $(fg)(x)=\sqrt{9+9x-x^2-x^3}$
(iv) $\frac{f}{g}:[-1,3[\to R$ defined by $\left(\frac{f}{g}\right)(x)=\sqrt{\frac{x+1}{9-x^2}}$
(v) $\frac{g}{f}:(-1,3]\to R$ defined by $\left(\frac{g}{f}\right)(x)=\sqrt{\frac{9-x^2}{x+1}}$

(vi) $2f-\sqrt{5}g:[-1,3]\to R$ defined by $(2f-\sqrt{5}g)(x)=2\sqrt{x+1}-\sqrt{45-5x^2}$
(vii) $f^2+7f:[-1,\infty)\to R$ defined by $(f^2+7f)(x)=x+1+7\sqrt{x+1}$
(viii) $\frac{5}{g}:(-3,3)\to R$ defined by $\left(\frac{5}{g}\right)(x)=\frac{5}{\sqrt{9-x^2}}$

5. (i) $f+g:(-\infty,1)\to R$ defined by $(f+g)(x)=\log_e(1-x)+[x]$
(ii) $fg:(-\infty,1)\to R$ defined by $(fg)(x)=[x]\log_e(1-x)$
(iii) $\frac{f}{g}:(-\infty,0)\to R$ defined by $\left(\frac{f}{g}\right)(x)=\frac{\log_e(1-x)}{[x]}$
(iv) $\frac{g}{f}:(-\infty,0)\cup(0,1)\to R$ defined by $\left(\frac{g}{f}\right)(x)=\frac{[x]}{\log_e(1-x)}$
$(f+g)(-1)=\log_e 2-1$
$(fg)(0)=0$, $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right)$ does not exist.
$\left(\frac{g}{f}\right)\left(\frac{1}{2}\right)=0$

6. 0, does not exist.

8. $f+g:R\to R$ defined by $(f+g)(x)=3x-2$
$f-g:R\to R$ defined by $(f-g)(x)=-x+4$
$\frac{f}{g}:R-\left\{\frac{3}{2}\right\}\to R$ defined by $\left(\frac{f}{g}\right)(x)=\frac{x+1}{2x-3}$

9. $f+g:[0,\infty)\to R$ defined by $(f+g)(x)=\sqrt{x}+x$
$f-g:[0,\infty)\to$ defined by $(f-g)(x)=\sqrt{x}-x$
$fg:[0,\infty)\to R$ defined by $(fg)(x)=x^{3/2}$
$\frac{f}{g}:(0,\infty)\to R$ defined by $\left(\frac{f}{g}\right)(x)=\frac{1}{\sqrt{x}}$

10. $(f+g)(x)=(x+1)^2$, $(f-g)(x)=x^2-2x-1$, $(fg)(x)=2x^3+x^2$, $\left(\frac{f}{g}\right)(x)=\frac{x^2}{2x+1}$

HINTS TO NCERT & SELECTED PROBLEMS

7. We have,

$$f(x)=\begin{cases}1-x, & x<0\\ 1, & x=0\\ x+1, & x>0\end{cases}$$

Let $f(x)=y$. Then,
$y=1-x$ or $x+y=1$ for $x<0$
$y=1$ for $x=1$
and $y=x+1$ or $-x+y=1$ for $x>0$.
So, the graph of $f(x)$ is as shown in Fig. 3.27.

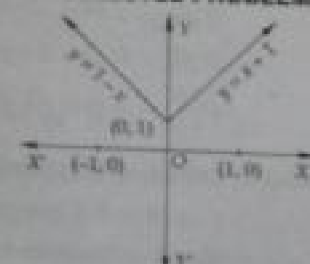


Fig. 3.27

8. $f:R\to R$ and $g:R\to R$ are given by $f(x)=x+1$ and $g(x)=2x-3$
Clearly, $D(f)=R$, $D(g)=R$. Therefore,
(i) $D(f+g)=D(f)\cap D(g)=R$
and, $(f+g)(x)=f(x)+g(x)=x+1+2x-3=3x-2$

(ii) $D(f-g) = D(f) \cap D(g) = R$

and, $(f-g)(x) = f(x) - g(x) = x+1-2x+3 = -x+4$

(iii) $D(fg) = D(f) \cap D(g) = R$

and, $(fg)(x) = f(x)g(x) = (x+1)(2x-3) = 2x^2 - x - 3$

(iv) $D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R - \left\{\frac{3}{2}\right\}$

and, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{2x-3}$

9. It is given that $f : [0, \infty) \to R$ and $g : R \to R$ such that $f(x) = \sqrt{x}$ and $g(x) = x$.

$D(f+g) = [0, \infty) \cap R = [0, \infty)$

So, $f+g : [0, \infty) \to R$ is given by

$(f+g)(x) = f(x) + g(x) = \sqrt{x} + x$

$D(f-g) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$

So, $f-g : [0, \infty) \to R$ is given by

$(f-g)(x) = f(x) - g(x) = \sqrt{x} - x$

$D(fg) = D(f) \cap D(g) = [0, \infty) \cap R = [0, \infty)$

So, $fg : [0, \infty) \to R$ is given by

$(fg)(x) = f(x)g(x) = \sqrt{x} \cdot x = x^{3/2}$

$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = (0, \infty)$

So, $\frac{f}{g} : (0, \infty) \to R$ is given by

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

10. We have,

$f(x) = x^2$ and $g(x) = 2x + 1$

Clearly, $D(f) = R$ and $D(g) = R$.

$\therefore$ $D(f \pm g) = D(f) \cap D(g) = R \cap R = R$

$D(fg) = D(f) \cap D(g) = R \cap R = R$

$D\left(\frac{f}{g}\right) = D(f) \cap D(g) - \{x : g(x) = 0\} = R \cap R - \left\{-\frac{1}{2}\right\} = R - \left\{-\frac{1}{2}\right\}$

$\therefore$ $f+g : R \to R$ is given by $(f+g)(x) = f(x) + g(x) = x^2 + 2x + 1$

$f-g : R \to R$ is given by $(f-g)(x) = f(x) - g(x) = x^2 - 2x - 1$

$fg : R \to R$ is given by $(fg)(x) = f(x)g(x) = x(2x+1)$

$\left(\frac{f}{g}\right) : R - \left\{\frac{1}{2}\right\} \to R$ is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{2x+1}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the range of the real function $f(x) = |x|$.

2. If f is a real function satisfying $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ for all $x \in R - \{0\}$, then write the expression for $f(x)$.
3. Write the range of the function $f(x) = \sin[x]$, where $\frac{-\pi}{4} \le x \le \frac{\pi}{4}$.
4. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where $[x]$ denotes the greatest integer less than or equal to x, then write the value of $f(\pi)$.
5. Write the range of the function $f(x) = \cos[x]$, where $\frac{-\pi}{2} < x < \frac{\pi}{2}$.
6. Write the range of the function $f(x) = e^{x-[x]}, x \in R$.
7. Let $f(x) = \frac{\alpha x}{x+1}, x \ne -1$. Then write the value of α satisfying $f(f(x)) = x$ for all $x \ne -1$.
8. If $f(x) = 1 - \frac{1}{x}$, then write the value of $f\left(f\left(\frac{1}{x}\right)\right)$.
9. Write the domain and range of the function $f(x) = \frac{x-2}{2-x}$.
10. If $f(x) = 4x - x^2, x \in R$, then write the value of $f(a+1) - f(a-1)$.
11. If f, g, h are real functions given by $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log_e x$, then write the value of $(hogof)\left(\sqrt{\frac{\pi}{4}}\right)$.
12. Write the domain and range of function $f(x)$ given by $f(x) = \frac{1}{\sqrt{x - |x|}}$.
13. Write the domain and range of $f(x) = \sqrt{x - [x]}$.
14. Write the domain and range of function $f(x)$ given by $f(x) = \sqrt{[x] - x}$.

ANSWERS

1. $[0, \infty)$ 2. $f(x) = x^2 - 2$, where $|x| \ge 2$ 3. $\{-\sin 1, 0, \sin 1\}$ 4. 0
5. $\{1, \cos 1, \cos 2\}$ 6. $[1, e)$ 7. $\alpha = -1$ 8. $\frac{x}{x-1}$
9. $D(f) = R - \{2\}, R(f) = \{-1\}$ 10. $4(2-a)$ 11. 0
12. $D(f) = \phi = R(f)$ 13. $D(f) = R, R(f) = [0, 1)$ 14. $D(f) = \phi = R(f)$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Let $A = \{1, 2, 3\}, B = \{2, 3, 4\}$, then which of the following is a function from A to B?
(a) $\{(1,2), (1,3), (2,3), (3,3)\}$ (b) $\{(1,3), (2,4)\}$
(c) $\{(1,3), (2,2), (3,3)\}$ (d) $\{(1,2), (2,3), (3,2), (3,4)\}$
2. If $f : Q \to Q$ is defined as $f(x) = x^2$, then $f^{-1}(9)$ is equal to
(a) 3 (b) -3 (c) $\{-3, 3\}$ (d) ϕ
3. Which one of the following is not a function?
(a) $\{(x, y) : x, y \in R, x^2 = y\}$ (b) $\{(x, y) : x, y \in R, y^2 = x\}$
(c) $\{(x, y) : x, y \in R, x = y^3\}$ (d) $\{(x, y) : x, y \in R, y = x^3\}$

4. If $f(x) = \cos(\log x)$, then $f(x^2)f(y^2) - \frac{1}{2}\left[f\left(\frac{x^2}{y^2}\right) + f(x^2y^2)\right]$ has the value
(a) -2 (b) -1 (c) $1/2$ (d) none of these

5. If $f(x) = \cos(\log x)$, then $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value
(a) -1 (b) $1/2$ (c) -2 (d) none of these

6. Let $f(x) = |x-1|$. Then,
(a) $f(x^2) = [f(x)]^2$
(b) $f(x+y) = f(x)f(y)$
(c) $f(|x|) = |f(x)|$
(d) none of these

7. The range of $f(x) = \cos[x]$, for $-\pi/2 < x < \pi/2$ is
(a) $\{-1, 1, 0\}$ (b) $\{\cos 1, \cos 2, 1\}$ (c) $\{\cos 1, -\cos 1, 1\}$ (d) $[-1, 1]$

8. Which of the following are functions ?
(a) $\{(x, y) : y^2 = x, x, y \in R\}$
(b) $\{(x, y) : y = |x|, x, y \in R\}$
(c) $\{(x, y) : x^2 + y^2 = 1, x, y \in R\}$
(d) $\{(x, y) : x^2 - y^2 = 1, x, y \in R\}$

9. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{3x + x^3}{1+3x^2}$, then $f(g(x))$ is equal to
(a) $f(3x)$ (b) $\{f(x)\}^3$ (c) $3f(x)$ (d) $-f(x)$

10. If $A = \{1, 2, 3\}$, $B = \{x, y\}$, then the number of functions that can be defined from A into B is
(a) 12 (b) 8 (c) 6 (d) 3

11. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to
(a) $\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) $2f(x)$ (d) $3f(x)$

12. If $f(x) = \cos(\log x)$, then value of $f(x)f(4) - \frac{1}{2}\left\{f\left(\frac{x}{4}\right) + f(4x)\right\}$, is
(a) 1 (b) -1 (c) 0 (d) ± 1

13. If $f(x) = \frac{2^x + 2^{-x}}{2}$, then $f(x+y)f(x-y)$ is equals to
(a) $\frac{1}{2}[f(2x) + f(2y)]$
(b) $\frac{1}{2}[f(2x) - f(2y)]$
(c) $\frac{1}{4}[f(2x) + f(2y)]$
(d) $\frac{1}{4}[f(2x) - f(2y)]$

14. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ $(x \neq 0)$, then $f(2)$ is equal to
(a) $-\frac{7}{4}$ (b) $\frac{5}{2}$ (c) -1 (d) none of these

15. Let $f : R \to R$ be defined by $f(x) = 2x + |x|$. Then $f(2x) + f(-x) - f(x) =$
(a) $2x$ (b) $2|x|$ (c) $-2x$ (d) $-2|x|$

16. If $f(x) = \log_e\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to
(a) $\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) $2f(x)$ (d) $3f(x)$

17. If $x \neq 1$ and $f(x) = \frac{x+1}{x-1}$ is a real function, then $f(f(f(2)))$ is
(a) 1 (b) 2 (c) 3 (d) 4

18. If $f(x) = \cos(\log_e x)$, then $f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right) - \frac{1}{2}\left\{f(xy) + f\left(\frac{x}{y}\right)\right\}$ is equal to
(a) $\cos(x-y)$ (b) $\log(\cos(x-y))$ (c) 1 (d) $\cos(x+y)$

19. Let $f(x) = x$, $g(x) = \frac{1}{x}$ and $h(x) = f(x)g(x)$. Then, $h(x) = 1$
(a) $x \in R$ (b) $x \in Q$ (c) $x \in R - Q$ (d) $x \in R, x \neq 0$

20. If $f(x) = \frac{\sin^4 x + \cos^2 x}{\sin^2 x + \cos^4 x}$ for $x \in R$, then $f(2002) =$
(a) 1 (b) 2 (c) 3 (d) 4

21. The function $f : R \to R$ is defined by $f(x) = \cos^2 x + \sin^4 x$. Then, $f(R) =$
(a) $[3/4, 1)$ (b) $(3/4, 1]$ (c) $[3/4, 1]$ (d) $(3/4, 1)$

22. Let $A = \{x \in R : x \neq 0, -4 \leq x \leq 4\}$ and $f : A \in R$ be defined by $f(x) = \frac{|x|}{x}$ for $x \in A$. Then th (is
(a) $\{1, -1\}$ (b) $\{x : 0 \leq x \leq 4\}$ (c) $\{1\}$ (d) $\{x : -4 \leq x \leq 0\}$

23. If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then the values of x such that $g(f(x)) = 8$ are
(a) 1, 2 (b) -1, 2 (c) $-1, -2$ (d) $1, -2$

24. If $f : [-2, 2] \to R$ is defined by $f(x) = \begin{cases} -1, & \text{for } -2 \leq x \leq 0 \\ x-1, & \text{for } 0 \leq x \leq 2 \end{cases}$, then
$\{x \in [-2, 2] : x \leq 0 \text{ and } f(|x|) = x\} =$
(a) $\{-1\}$ (b) $\{0\}$ (c) $\left\{-\frac{1}{2}\right\}$ (d) ϕ

25. If $e^{f(x)} = \frac{10+x}{10-x}$, $x \in (-10, 10)$ and $f(x) = kf\left(\frac{200x}{100+x^2}\right)$, then $k =$
(a) 0.5 (b) 0.6 (c) 0.7 (d) 0.8

26. If f is a real valued function given by $f(x) = 27x^3 + \frac{1}{x^3}$ and α, β are roots of $3x + \frac{1}{x} = 12$. Then,
(a) $f(\alpha) \neq f(\beta)$ (b) $f(\alpha) = 10$ (c) $f(\beta) = -10$ (d) none of these

27. If $f(x) = 64x^3 + \frac{1}{x^3}$ and α, β are the roots of $4x + \frac{1}{x} = 3$. Then,
(a) $f(\alpha) = f(\beta) = -9$
(b) $f(\alpha) = f(\beta) = 63$
(c) $f(\alpha) \neq f(\beta)$
(d) none of these

28. If $3f(x) + 5f\left(\frac{1}{x}\right) = \frac{1}{x} - 3$ for all non-zero x, then $f(x) =$
(a) $\frac{1}{14}\left(\frac{3}{x} + 5x - 6\right)$
(b) $\frac{1}{14}\left(-\frac{3}{x} + 5x - 6\right)$

5

TRIGONOMETRIC FUNCTIONS

5.1 INTRODUCTION

In the present chapter, we will first introduce trigonometric ratios which are also known as trigonometric functions and then the identities involving them.

5.2 TRIGONOMETRIC RATIOS OR FUNCTIONS

Consider an angle $\theta = \angle XOA$ as shown in Fig. 5.1. Let P be any point other than O on its terminal side OA and let PM be perpendicular from P on x-axis. Let length $OP = r$, $OM = x$ and $MP = y$. We take the length $OP = r$ always positive while x and y can be positive or negative depending upon the position of the terminal side OA of $\angle XOA$.

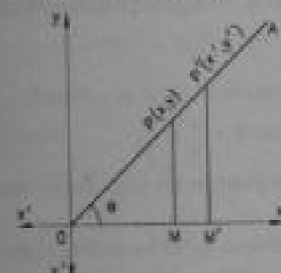


Fig. 5.1

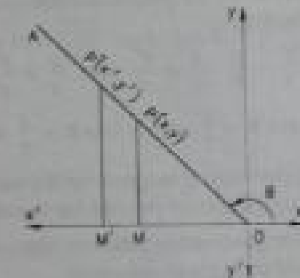


Fig. 5.2

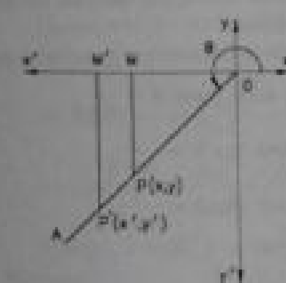


Fig. 5.3

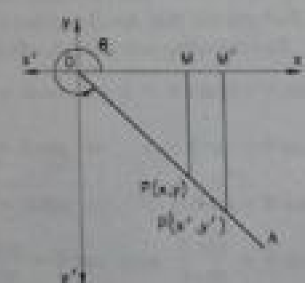


Fig. 5.4

In the right angled triangle OMP, we have

Base $= OM = x$, Perpendicular $= PM = y$ and, Hypotenues $= OP = r$

We define the following trigonometric ratios which are also known as trgonometric functions.

$$\text{Sine } \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{y}{r}, \text{ and is written as } \sin\theta;$$

$$\text{Cosine } \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{x}{r}, \text{ and is written as } \cos\theta;$$

$$\text{Tangent } \theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{y}{x}, \text{ and is written as } \tan\theta;$$

$$\text{Cosecant } \theta = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{r}{y}, \text{ and is written as } \operatorname{cosec}\theta;$$

$$\text{Secant } \theta = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{r}{x}, \text{ and is written as } \sec\theta;$$

$$\text{Cotangent } \theta = \frac{\text{Base}}{\text{Perpendicular}} = \frac{x}{y}, \text{ and is written as } \cot\theta.$$

NOTE 1 *The triangle OMP is known as the triangle of reference.*

NOTE 2 *It should be noted that sin θ does not mean the product of sin and θ. The sin θ is correctly read as sine of angle θ.*

These functions depend only on the value of the angle θ and not on the position of the point P chosen on the terminal side of the angle θ as shown in the following theorem.

THEOREM *The trigonometric ratios are same for the same angle.*

PROOF Let $P'(x', y')$ be any point other than $P(x, y)$ on the terminal side OA with $OP' = r'$. Let $P'M'$ be perpendicular on x-axis. Clearly, triangles OMP and $OM'P'$ are similar. Therefore,

$$\frac{y}{r} = \frac{y'}{r'}, \frac{x}{r} = \frac{x'}{r'} \text{ and } \frac{y}{x} = \frac{y'}{x'}$$

$$\Rightarrow \quad \sin\theta = \frac{y}{r} = \frac{y'}{r'}, \cos\theta = \frac{x}{r} = \frac{x'}{r'} \text{ and } \tan\theta = \frac{y}{x} = \frac{y'}{x'}$$

Thus, sine, cosine and tangent are the same whatever point be taken on the terminal side OA. Similarly, it can be proved for the other ratios.

Hence, the trigonometric ratios or trigonometric functions are independent of the choice of the size of triangle of reference.

Q.E.D.

REMARK 1 *If the terminal side coincides with x-axis, then cosec θ and cot θ are not defined. If it coincides with y-axis, then sec θ and tan θ are not defined.*

REMARK 2 *The following relations are obvious from the definitions of trigonometric ratios:*

(i) $\sin\theta \times \operatorname{cosec}\theta = 1 \Rightarrow \sin\theta = \frac{1}{\operatorname{cosec}\theta}$ and $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

(ii) $\cos\theta \times \sec\theta = 1 \Rightarrow \cos\theta = \frac{1}{\sec\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$

(iii) $\tan\theta \times \cot\theta = 1 \Rightarrow \cot\theta = \frac{1}{\tan\theta}$ and $\tan\theta = \frac{1}{\cot\theta}$

(iv) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$

REMARK 3 *The trigonometric ratios may be positive or negative depending upon x and/or y as discussed in section 5.4.*

5.3 TRIGONOMETRIC IDENTITIES

DEFINITION *An equation involving trigonometric functions which is true for all those angles for which the functions are defined is called a trigonometric identity.*

ILLUSTRATION 1 $\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ *etc. are trigonometric identities as they hold for all those values of θ except those values for which sec θ and cosec θ are not defined.*

ILLUSTRATION 2 *The equation* $\sin\theta = \cos\theta$ *is a trigonometric equation but not a trigonometric identity because it does not hold for all values of θ.*

5.3.1 FUNDAMENTAL TRIGONOMETRIC IDENTITIES

The following are some fundamental identities:

THEOREM Prove that:

(i) $\sin\theta = \frac{1}{\operatorname{cosec}\theta}$ or $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$

(ii) $\cos\theta = \frac{1}{\sec\theta}$ or $\sec\theta = \frac{1}{\cos\theta}$

(iii) $\cot\theta = \frac{1}{\tan\theta}$ or $\tan\theta = \frac{1}{\cot\theta}$

(iv) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

(v) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(vi) $\sin^2\theta + \cos^2\theta = 1$

(vii) $1 + \tan^2\theta = \sec^2\theta$

(viii) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

PROOF Let a revolving ray start from OX and revolve into the position OP to trace out any angle θ in any of the four quadrants. From P drawn PM perpendicular to x-axis. In the right angled triangle OMP, we have

$$OP^2 = OM^2 + PM^2, \sin\theta = \frac{PM}{OP}, \cos\theta = \frac{OM}{OP} \text{ and } \tan\theta = \frac{PM}{OM}$$

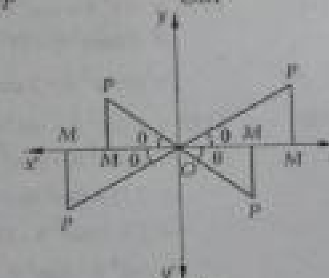


Fig. 5.5

Clearly, identities in (i) to (v) are trivial.

(vi) We have,

$$\sin^2\theta + \cos^2\theta$$
$$= \left(\frac{PM}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2$$
$$= \frac{PM^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1$$

Hence, $\sin^2\theta + \cos^2\theta = 1$

(vii) We have,

$$1 + \tan^2\theta = 1 + \left(\frac{PM}{OM}\right)^2$$

$$\Rightarrow \quad 1 + \tan^2\theta = 1 + \frac{PM^2}{OM^2} = \frac{OM^2 + PM^2}{OM^2} = \frac{OP^2}{OM^2} = \left(\frac{OP}{OM}\right)^2 = \sec^2\theta$$

Hence, $1 + \tan^2\theta = \sec^2\theta$

(viii) We have,

$$1 + \cot^2\theta = 1 + \left(\frac{OM}{PM}\right)^2$$

$$\Rightarrow \quad 1 + \cot^2\theta = 1 + \frac{OM^2}{PM^2} = \frac{PM^2 + OM^2}{PM^2} = \frac{OP^2}{PM^2} = \left(\frac{OP}{PM}\right)^2 = \operatorname{cosec}^2\theta$$

Hence, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Q.E.D.

NOTE *It should be noted that* $(\sin\theta)^2$ *is written as* $\sin^2\theta$, $(\cos\theta)^2$ *is written as* $\cos^2\theta$ *etc.*

We shall now discuss more identities involving the trigonometrical functions in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Prove the following identities:*

(i) $\sin^8\theta - \cos^8\theta = (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta)$

(ii) $\cot^4\theta + \cot^2\theta = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta$

(iii) $2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \cot^4\theta - \tan^4\theta$

(iv) $(\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2 = \tan^2\theta + \cot^2\theta + 7$

SOLUTION

(i) $\text{LHS} = (\sin^8\theta - \cos^8\theta) = (\sin^4\theta)^2 - (\cos^4\theta)^2$
$= (\sin^4\theta - \cos^4\theta)(\sin^4\theta + \cos^4\theta)$
$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta)$
$= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta)$
$= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta)$
$= (\sin^2\theta - \cos^2\theta)[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$
$= (\sin^2\theta - \cos^2\theta)(1 - 2\sin^2\theta\cos^2\theta) = \text{RHS}$

(ii) $\text{LHS} = \cot^4\theta + \cot^2\theta = (\cot^2\theta)^2 + \cot^2\theta$
$= (\operatorname{cosec}^2\theta - 1)^2 + (\operatorname{cosec}^2\theta - 1)$ $[\because 1 + \cot^2\theta = \operatorname{cosec}^2\theta]$
$= \operatorname{cosec}^4\theta - 2\operatorname{cosec}^2\theta + 1 + \operatorname{cosec}^2\theta - 1 = \operatorname{cosec}^4\theta - \operatorname{cosec}^2\theta = \text{RHS}$

(iii) $\text{LHS} = 2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta$
$= 2\sec^2\theta - (\sec^2\theta)^2 - 2\operatorname{cosec}^2\theta + (\operatorname{cosec}^2\theta)^2$
$= 2(1 + \tan^2\theta) - (1 + \tan^2\theta)^2 - 2(1 + \cot^2\theta) + (\cot^2\theta + 1)^2$
$= 2 + 2\tan^2\theta - (1 + \tan^4\theta + 2\tan^2\theta) - 2 - 2\cot^2\theta + (\cot^4\theta - 2\cot^2\theta + 1)$
$= \cot^4\theta - \tan^4\theta = \text{RHS}$

(iv) $\text{LHS} = (\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$
$= \sin^2\theta + \operatorname{cosec}^2\theta + 2\sin\theta\operatorname{cosec}\theta + \cos^2\theta + \sec^2\theta + 2\cos\theta\sec\theta$
$= (\sin^2\theta + \cos^2\theta) + (\operatorname{cosec}^2\theta + \sec^2\theta) + 2 + 2$
$= 1 + 1 + \cot^2\theta + 1 + \tan^2\theta + 4 = \tan^2\theta + \cot^2\theta + 7 = \text{RHS}$

EXAMPLE 2 *Prove the following identities:*

(i) $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$

(ii) $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}$

SOLUTION

(i) $\text{LHS} = (1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta)$
$= \left(1 + \dfrac{\cos\theta}{\sin\theta} - \dfrac{1}{\sin\theta}\right)\left(1 + \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta}\right)$
$= \dfrac{(\sin\theta + \cos\theta - 1)}{\sin\theta}\,\dfrac{(\sin\theta + \cos\theta + 1)}{\cos\theta}$
$= \dfrac{(\sin\theta + \cos\theta)^2 - 1}{\sin\theta\cos\theta} = \dfrac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$

$= \dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}$

(ii) $\text{LHS} = \dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$
$= \dfrac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$ $[\because \sec^2\theta - \tan^2\theta = 1]$
$= \dfrac{(\sec\theta + \tan\theta)[1 - (\sec\theta - \tan\theta)]}{\tan\theta - \sec\theta + 1} = \dfrac{(\sec\theta + \tan\theta)(\tan\theta - \sec\theta + 1)}{\tan\theta - \sec\theta + 1}$
$= \sec\theta + \tan\theta = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1 + \sin\theta}{\cos\theta} = \text{RHS}$

EXAMPLE 3 *If* $\tan\theta + \sin\theta = m$ *and* $\tan\theta - \sin\theta = n$, *show that* $m^2 - n^2 = 4\sqrt{mn}$.

SOLUTION We have,

$\text{LHS} = m^2 - n^2 = (\tan\theta + \sin\theta)^2 - (\tan\theta - \sin\theta)^2 = 4\tan\theta\sin\theta$ ...(i)

$\text{RHS} = 4\sqrt{mn} = 4\sqrt{(\tan\theta + \sin\theta)(\tan\theta - \sin\theta)} = 4\sqrt{\tan^2\theta - \sin^2\theta}$

$\Rightarrow \text{RHS} = 4\sqrt{\dfrac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta} = 4\sqrt{\dfrac{\sin^2\theta - \sin^2\theta\cos^2\theta}{\cos^2\theta}} = 4\sqrt{\dfrac{\sin^2\theta(1 - \cos^2\theta)}{\cos^2\theta}}$

$\Rightarrow \text{RHS} = 4\sqrt{\dfrac{\sin^4\theta}{\cos^2\theta}} = 4\dfrac{\sin^2\theta}{\cos\theta} = 4\tan\theta\sin\theta$ ...(ii)

From (i) and (ii), we have LHS = RHS.

EXAMPLE 4 *If* $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, *prove that* $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

SOLUTION We have,

$\cos\theta + \sin\theta = \sqrt{2}\cos\theta$
$\Rightarrow (\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$
$\Rightarrow \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$
$\Rightarrow \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$
$\Rightarrow (\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$
$\Rightarrow \cos\theta - \sin\theta = \dfrac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta}$
$\Rightarrow \cos\theta - \sin\theta = \dfrac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta}$ $[\because \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$
$\Rightarrow \cos\theta - \sin\theta = \sqrt{2}\sin\theta$

EXAMPLE 5 *If* $a\cos\theta + b\sin\theta = x$ *and* $a\sin\theta - b\cos\theta = y$, *prove that* $a^2 + b^2 = x^2 + y^2$.

SOLUTION We have,

$\text{RHS} = x^2 + y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$
$\Rightarrow \text{RHS} = (a^2\cos^2\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta) + (a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta)$
$\Rightarrow \text{RHS} = a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2 = \text{LHS}$

EXAMPLE 6 *If* $\dfrac{ax}{\cos\theta} + \dfrac{by}{\sin\theta} = a^2 - b^2$ *and* $\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0$, *prove that*

$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

SOLUTION We have,

$\dfrac{ax\sin\theta}{\cos^2\theta} - \dfrac{by\cos\theta}{\sin^2\theta} = 0$

$\Rightarrow ax\sin^3\theta - by\cos^3\theta = 0 \Rightarrow \frac{\sin^3\theta}{by} = \frac{\cos^3\theta}{ax}$

$\Rightarrow \left(\frac{\sin^3\theta}{by}\right)^{2/3} = \left(\frac{\cos^3\theta}{ax}\right)^{2/3}$

$\Rightarrow \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}}$

$\Rightarrow \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}} = \frac{\sin^2\theta + \cos^2\theta}{(by)^{2/3} + (ax)^{2/3}}$ [Using ratio and proportion]

$\Rightarrow \frac{\sin^2\theta}{(by)^{2/3}} = \frac{\cos^2\theta}{(ax)^{2/3}} = \frac{1}{(ax)^{2/3} + (by)^{2/3}}$

$\Rightarrow \sin^2\theta = \frac{(by)^{2/3}}{(ax)^{2/3} + (by)^{2/3}}$ and $\cos^2\theta = \frac{(ax)^{2/3}}{(ax)^{2/3} + (by)^{2/3}}$

$\Rightarrow \sin\theta = \frac{(by)^{1/3}}{\sqrt{(ax)^{2/3} + (by)^{2/3}}}$ and $\cos\theta = \frac{(ax)^{1/3}}{\sqrt{(ax)^{2/3} + (by)^{2/3}}}$

Substituting these values in $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 - b^2$, we get

$$(ax)^{2/3}\sqrt{(ax)^{2/3} + (by)^{2/3}} + (by)^{2/3}\sqrt{(ax)^{2/3} + (by)^{2/3}} = a^2 - b^2$$

$\Rightarrow \left[\sqrt{(ax)^{2/3} + (by)^{2/3}}\right]\left[(ax)^{2/3} + (by)^{2/3}\right] = a^2 - b^2$

$\Rightarrow \left[(ax)^{2/3} + (by)^{2/3}\right]^{3/2} = a^2 - b^2$

$\Rightarrow (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

EXAMPLE 7 *If* $m^2 + m'^2 + 2mm'\cos\theta = 1$, $n^2 + n'^2 + 2nn'\cos\theta = 1$, *and* $mn + m'n' + (mn' + m'n)\cos\theta = 0$, *prove that* $m^2 + n^2 = \operatorname{cosec}^2\theta$.

SOLUTION We have,

$$m^2 + m'^2 + 2mm'\cos\theta = 1 \text{ and } n^2 + n'^2 + 2nn'\cos\theta = 1$$

$\Rightarrow m'^2 + 2mm'\cos\theta + m^2\cos^2\theta - m^2\cos^2\theta + m^2 = 1$

And, $n'^2 + 2nn'\cos\theta + n^2\cos^2\theta - n^2\cos^2\theta + n^2 = 1$

$\Rightarrow (m' + m\cos\theta)^2 + m^2(1 - \cos^2\theta) = 1$ and $(n' + n\cos\theta)^2 + n^2(1 - \cos^2\theta) = 1$

$\Rightarrow (m' + m\cos\theta)^2 = 1 - m^2\sin^2\theta$ ...(i)

And, $(n' + n\cos\theta)^2 = 1 - n^2\sin^2\theta$ ...(ii)

Now, $(m' + m\cos\theta)(n' + n\cos\theta)$

$= m'n' + (mn' + m'n)\cos\theta + mn\cos^2\theta$

$= -mn + mn\cos^2\theta$ [$\because mn + m'n' + (mn' + m'n)\cos\theta = 0$]

$= -mn(1 - \cos^2\theta) = -mn\sin^2\theta$

$\therefore (m' + m\cos\theta)^2(n' + n\cos\theta)^2 = m^2n^2\sin^4\theta$ [On squaring both sides]

$\Rightarrow (1 - m^2\sin^2\theta)(1 - n^2\sin^2\theta) = m^2n^2\sin^4\theta$ [Using (i) and (ii)]

$\Rightarrow 1 - (m^2 + n^2)\sin^2\theta + m^2n^2\sin^4\theta = m^2n^2\sin^4\theta$

$\Rightarrow 1 = (m^2 + n^2)\sin^2\theta$

$\Rightarrow m^2 + n^2 = \operatorname{cosec}^2\theta$

EXAMPLE 8 *If* $a\cos\theta - b\sin\theta = c$, *show that* $a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

SOLUTION We have,

$(a\cos\theta - b\sin\theta)^2 + (a\sin\theta + b\cos\theta)^2$

$= a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) - 2ab\sin\theta\cos\theta + 2ab\sin\theta\cos\theta$

$= a^2 + b^2$

$\therefore (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - (a\cos\theta - b\sin\theta)^2$

$\Rightarrow (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$ [$\because a\cos\theta - b\sin\theta = c$]

$\Rightarrow a\sin\theta + b\cos\theta = \pm\sqrt{a^2 + b^2 - c^2}$

EXAMPLE 9 *Given that*

$$(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma) = (1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma).$$

Show that one of the values of each member of this equality is $\sin\alpha\sin\beta\sin\gamma$.

SOLUTION We have,

$$(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma) = (1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma)$$

Multiplying both sides by $(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma)$, we get

$$(1 + \cos\alpha)^2(1 + \cos\beta)^2(1 + \cos\gamma)^2 = (1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma)(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma)$$

$\Rightarrow (1 + \cos\alpha)^2(1 + \cos\beta)^2(1 + \cos\gamma)^2 = (1 - \cos^2\alpha)(1 - \cos^2\beta)(1 - \cos^2\gamma)$

$\Rightarrow (1 + \cos\alpha)^2(1 + \cos\beta)^2(1 + \cos\gamma)^2 = \sin^2\alpha\sin^2\beta\sin^2\gamma$

$\Rightarrow (1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma) = \pm\sin\alpha\sin\beta\sin\gamma$

Hence, one of the values of $(1 + \cos\alpha)(1 + \cos\beta)(1 + \cos\gamma)$ is $\sin\alpha\sin\beta\sin\gamma$.

Similarly, by multiplying both sides by $(1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma)$, we find tha one of the values of $(1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma)$ is $\sin\alpha\sin\beta\sin\gamma$.

EXAMPLE 10 *If* $\frac{\sin A}{\sin B} = p$ *and* $\frac{\cos A}{\cos B} = q$, *find* $\tan A$ *and* $\tan B$.

SOLUTION We have,

$\frac{\sin A}{\sin B} = p$ and $\frac{\cos A}{\cos B} = q$

$\Rightarrow \frac{\sin A}{\sin B}\cdot\frac{\cos B}{\cos A} = \frac{p}{q}$

$\Rightarrow \frac{\tan A}{\tan B} = \frac{p}{q}$

$\Rightarrow \frac{\tan A}{p} = \frac{\tan B}{q} = \lambda$ (say)

$\Rightarrow \tan A = p\lambda$ and $\tan B = q\lambda$ ...(i)

Now, $\sin A = p\sin B$

$\Rightarrow \frac{\tan A}{\sqrt{1 + \tan^2 A}} = p\frac{\tan B}{\sqrt{1 + \tan^2 B}}$

$\Rightarrow \frac{p\lambda}{\sqrt{1 + p^2\lambda^2}} = p\cdot\frac{q\lambda}{\sqrt{1 + q^2\lambda^2}}$

$\Rightarrow p^2(1 + q^2\lambda^2) = p^2q^2(1 + p^2\lambda^2)$

$\Rightarrow \lambda^2(q^2 - p^2q^2) = q^2 - 1$

$$\Rightarrow \quad \lambda^2 = \frac{q^2-1}{q^2(1-p^2)}$$

$$\Rightarrow \quad \lambda = \pm\frac{1}{q}\sqrt{\frac{q^2-1}{1-p^2}}$$

$$\therefore \quad \tan A = \pm\frac{p}{q}\sqrt{\frac{q^2-1}{1-p^2}} \text{ and } \tan B = \pm\sqrt{\frac{q^2-1}{1-p^2}}$$ [Using (i

EXAMPLE 11 *If $a\cos^3\theta + 3a\cos\theta\sin^2\theta = m$ and $a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$, then pro… that*

$$(m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$$

SOLUTION We have,

$$a\cos^3\theta + 3a\cos\theta\sin^2\theta = m \text{ and } a\sin^3\theta + 3a\cos^2\theta\sin\theta = n$$

$$\Rightarrow \quad a\cos^3\theta + 3a\cos\theta\sin^2\theta + a\sin^3\theta + 3a\cos^2\theta\sin\theta = m+n$$

And, $$a\cos^3\theta + 3a\cos\theta\sin^2\theta - a\sin^3\theta - 3a\cos^2\theta\sin\theta = m-n$$

$$\Rightarrow \quad a(\cos\theta+\sin\theta)^3 = m+n \text{ and } a(\cos\theta-\sin\theta)^3 = m-n$$

$$\Rightarrow \quad \cos\theta+\sin\theta = \left(\frac{m+n}{a}\right)^{1/3} \text{ and } \cos\theta-\sin\theta = \left(\frac{m-n}{a}\right)^{1/3}$$

$$\Rightarrow \quad (\cos\theta+\sin\theta)^2 + (\cos\theta-\sin\theta)^2 = \left(\frac{m+n}{a}\right)^{2/3} + \left(\frac{m-n}{a}\right)^{2/3}$$

$$\Rightarrow \quad 2(\cos^2\theta+\sin^2\theta) = \frac{(m+n)^{2/3}}{a^{2/3}} + \frac{(m-n)^{2/3}}{a^{2/3}}$$

$$\Rightarrow \quad (m+n)^{2/3} + (m-n)^{2/3} = 2a^{2/3}$$

EXAMPLE 12 *If $\sec\theta + \tan\theta = p$, obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of p.*

SOLUTION We know that: $\sec^2\theta - \tan^2\theta = 1$

$$\Rightarrow \quad (\sec\theta+\tan\theta)(\sec\theta-\tan\theta) = 1$$

$$\Rightarrow \quad p(\sec\theta-\tan\theta) = 1$$

$$\Rightarrow \quad \sec\theta - \tan\theta = \frac{1}{p}$$

Now, $\sec\theta+\tan\theta = p$ and $\sec\theta-\tan\theta = \frac{1}{p}$

$$\Rightarrow \quad (\sec\theta+\tan\theta)+(\sec\theta-\tan\theta) = p+\frac{1}{p}$$

$$\Rightarrow \quad 2\sec\theta = p+\frac{1}{p} \Rightarrow \sec\theta = \frac{p^2+1}{2p}$$

Again, $\sec\theta+\tan\theta = p$ and $\sec\theta-\tan\theta = \frac{1}{p}$

$$\Rightarrow \quad (\sec\theta+\tan\theta)-(\sec\theta-\tan\theta) = p-\frac{1}{p}$$

$$\Rightarrow \quad 2\tan\theta = p-\frac{1}{p} \Rightarrow \tan\theta = \frac{p^2-1}{2p}$$

Dividing (ii) by (i), we get $\sin\theta = \frac{p^2-1}{p^2+1}$

EXAMPLE 13 *If $\tan^2\theta = 1-a^2$, prove that $\sec\theta + \tan^3\theta\operatorname{cosec}\theta = (2-a^2)^{3/2}$. Also, find the values of a for which the above result holds true.*

SOLUTION We have,

$$\sec\theta + \tan^3\theta\operatorname{cosec}\theta$$

$$= \sec\theta\left\{1+\tan^3\theta\frac{\operatorname{cosec}\theta}{\sec\theta}\right\}$$

$$= \sqrt{1+\tan^2\theta}\,\{1+\tan^3\theta\times\cot\theta\}$$

$$= (1+\tan^2\theta)^{3/2}$$

$$= (1+1-a^2)^{3/2} \qquad [\because \tan^2\theta = 1-a^2]$$

$$= (2-a^2)^{3/2}$$

Now,

$$\tan^2\theta \geq 0 \text{ for all } \theta$$

$$\Rightarrow \quad 1-a^2 \geq 0 \Rightarrow a^2-1 \leq 0 \Rightarrow -1 \leq a \leq 1 \qquad \text{...(i)}$$

Also, LHS of $\sec\theta + \tan^3\theta\operatorname{cosec}\theta = (2-a^2)^{3/2}$ is real for all $\theta \in R$.

So, RHS must also be real.

$$\therefore \quad 2-a^2 \geq 0 \Rightarrow a^2-2 \leq 0 \Rightarrow -\sqrt{2} \leq a \leq \sqrt{2} \qquad \text{...(ii)}$$

From (i) and (ii), we find that the given relation holds true for all $a \in [-1, 1]$.

EXAMPLE 14 *Prove that :*

$$2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta = \frac{1-\tan^8\theta}{\tan^4\theta}$$

SOLUTION We have,

$$2\sec^2\theta - \sec^4\theta - 2\operatorname{cosec}^2\theta + \operatorname{cosec}^4\theta$$

$$= 2(1+\tan^2\theta) - (1+\tan^2\theta)^2 - 2(1+\cot^2\theta) + (1+\cot^2\theta)^2$$

$$= 2(1+\tan^2\theta - 1 - \cot^2\theta) + (1+2\cot^2\theta+\cot^4\theta) - (1+2\tan^2\theta+\tan^4\theta)$$

$$= 2(\tan^2\theta - \cot^2\theta) + (2\cot^2\theta - 2\tan^2\theta) + \cot^4\theta - \tan^4\theta$$

$$= \frac{1}{\tan^4\theta} - \tan^4\theta = \frac{1-\tan^8\theta}{\tan^4\theta}$$

EXAMPLE 15 *Prove that :*

$$3(\sin\theta-\cos\theta)^4 + 6(\sin\theta+\cos\theta)^2 + 4(\sin^6\theta+\cos^6\theta) - 13 = 0$$

SOLUTION We have,

$$3(\sin\theta-\cos\theta)^4 + 6(\sin\theta+\cos\theta)^2 + 4(\sin^6\theta+\cos^6\theta) - 13$$

$$= 3\{(\sin\theta-\cos\theta)^2\}^2 + 6(\sin\theta+\cos\theta)^2 + 4\{(\sin^2\theta+\cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)\} - 13$$

$$= 3(1-2\sin\theta\cos\theta)^2 + 6(1+2\sin\theta\cos\theta) + 4(1-3\sin^2\theta\cos^2\theta) - 13$$

$$= 3(1-4\sin\theta\cos\theta+4\sin^2\theta\cos^2\theta) + 6(1+2\sin\theta\cos\theta) + 4(1-3\sin^2\theta\cos^2\theta) - 13$$

$$= 3+6+4-13$$

$$= 0$$

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Hence, $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \dfrac{2-\frac{4}{3}}{1-2\times\frac{-4}{3}} = \dfrac{2}{11}$

Now, $\pi < \alpha < \dfrac{3\pi}{2}$ and $\dfrac{\pi}{2} < \beta < \pi \Rightarrow \dfrac{3\pi}{2} < \alpha+\beta < \dfrac{5\pi}{2}$

We know that tangent function is positive in I and III quadrants.

$\therefore \quad \dfrac{3\pi}{2} < \alpha+\beta < \dfrac{5\pi}{2}$ and $\tan(\alpha+\beta) = \dfrac{2}{11} > 0 \Rightarrow \alpha+\beta$ lies in I quadrant.

Type II ON FINDING THE TRIGONOMETRIC RATIOS OF ANGLES WHICH ARE MULTIPLES OF 15°

EXAMPLE 5 *Find the values of the following:*

(i) $\sin 75°$ [NCERT] (ii) $\cos 75°$ [NCERT] (iii) $\sin 15°$ (iv) $\cos 15°$

SOLUTION We have,

(i) $\sin 75° = \sin(45°+30°) = \sin 45°\cos 30° + \cos 45°\sin 30°$

$\Rightarrow \sin 75° = \dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}}\times\dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

(ii) $\cos 75° = \cos(45°+30°) = \cos 45°\cos 30° - \sin 45°\sin 30°$

$\Rightarrow \cos 75° = \dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}}\times\dfrac{1}{2} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(iii) $\sin 15° = \sin(45°-30°) = \sin 45°\cos 30° - \cos 45°\sin 30°$

$\Rightarrow \sin 15° = \dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2} - \dfrac{1}{\sqrt{2}}\times\dfrac{1}{2} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos 15° = \cos(45°-30°) = \cos 45°\cos 30° + \sin 45°\sin 30°$

$\Rightarrow \cos 15° = \dfrac{1}{\sqrt{2}}\times\dfrac{\sqrt{3}}{2} + \dfrac{1}{\sqrt{2}}\times\dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

EXAMPLE 6 *Find the values of the following:*

(i) $\tan 15°$ [NCERT] (ii) $\tan 75°$ (iii) $\tan 105°$ (iv) $\tan\dfrac{13\pi}{12}$ [NCERT]

SOLUTION We have,

(i) $\tan 15° = \tan(45°-30°) = \dfrac{\tan 45° - \tan 30°}{1+\tan 45°\tan 30°} = \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

(ii) $\tan 75° = \tan(45°+30°) = \dfrac{\tan 45° + \tan 30°}{1-\tan 45°\tan 30°} = \dfrac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

(iii) $\tan 105° = \tan(90°+15°) = -\cot 15° = -\dfrac{1}{\tan 15°} = -\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ [Using (i)]

(iv) $\tan\dfrac{13\pi}{12} = \tan\left(\pi+\dfrac{\pi}{12}\right) = \tan\dfrac{\pi}{12}$ [$\because \tan(\pi+\theta) = \tan\theta$]

$= \tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ [Using (i)]

EXAMPLE 7 *Prove that* $\tan 75° + \cot 75° = 4$

SOLUTION We have,

$\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ [See Ex. 6]

$\therefore \cot 75° = \dfrac{1}{\tan 75°} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

Now,

$$\text{LHS} = \tan 75° + \cot 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{(4+2\sqrt{3})+(4-2\sqrt{3})}{3-1} = \frac{8}{2} = 4 = \text{RHS}$$

Type III ON USING THE FOLLOWING FORMULAE:

(i) $\sin A\cos B \pm \cos A\sin B = \sin(A\pm B)$ (ii) $\cos A\cos B \pm \sin A\sin B = \cos(A\mp B)$

EXAMPLE 8 *Evaluate the following:*

(i) $\sin\dfrac{7\pi}{12}\cos\dfrac{\pi}{4} - \cos\dfrac{7\pi}{12}\sin\dfrac{\pi}{4}$ (ii) $\sin\dfrac{\pi}{4}\cos\dfrac{\pi}{12} + \cos\dfrac{\pi}{4}\sin\dfrac{\pi}{12}$

(iii) $\cos\dfrac{2\pi}{3}\cos\dfrac{\pi}{4} - \sin\dfrac{2\pi}{3}\sin\dfrac{\pi}{4}$

SOLUTION We have,

(i) $\sin\dfrac{7\pi}{12}\cos\dfrac{\pi}{4} - \cos\dfrac{7\pi}{12}\sin\dfrac{\pi}{4}$

$= \sin\left(\dfrac{7\pi}{12}-\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ [$\because \sin A\cos B - \cos A\sin B = \sin(A-B)$]

(ii) $\sin\dfrac{\pi}{4}\cos\dfrac{\pi}{12} + \cos\dfrac{\pi}{4}\sin\dfrac{\pi}{12} = \sin\left(\dfrac{\pi}{4}+\dfrac{\pi}{12}\right) = \sin\dfrac{4\pi}{12} = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

(iii) $\cos\dfrac{2\pi}{3}\cos\dfrac{\pi}{4} - \sin\dfrac{2\pi}{3}\sin\dfrac{\pi}{4}$

$= \cos\left(\dfrac{2\pi}{3}+\dfrac{\pi}{4}\right) = \cos\dfrac{11\pi}{12}$

$= \cos 165° = \cos(180°-15°) = -\cos 15° = -\dfrac{\sqrt{3}+1}{2\sqrt{2}}$ [See Ex. 5 (iv)]

EXAMPLE 9 *Prove that:*

$\cos\left(\dfrac{\pi}{4}-A\right)\cos\left(\dfrac{\pi}{4}-B\right) - \sin\left(\dfrac{\pi}{4}-A\right)\sin\left(\dfrac{\pi}{4}-B\right) = \sin(A+B)$ [NCERT]

SOLUTION We have,

$\text{LHS} = \cos\left(\dfrac{\pi}{4}-A\right)\cos\left(\dfrac{\pi}{4}-B\right) - \sin\left(\dfrac{\pi}{4}-A\right)\sin\left(\dfrac{\pi}{4}-B\right)$

$= \cos\left\{\left(\dfrac{\pi}{4}-A\right)+\left(\dfrac{\pi}{4}-B\right)\right\} = \cos\left\{\dfrac{\pi}{2}-(A+B)\right\}$

$= \sin(A+B) = \text{RHS}$

EXAMPLE 10 *Prove that:*

$\sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A = \cos A$ [NCERT]

SOLUTION We have,

$$\text{LHS} = \sin(n+1)A\sin(n+2)A + \cos(n+1)A\cos(n+2)A$$
$$= \cos(n+2)A\cos(n+1)A + \sin(n+2)A\sin(n+1)A$$
$$= \cos\{(n+2)A-(n+1)A\} = \cos A = \text{RHS}$$

EXAMPLE 11 *If* $3\tan A\tan B = 1$, *prove that* $2\cos(A+B) = \cos(A-B)$

SOLUTION We have,

$$3\tan A\tan B = 1$$
$$\Rightarrow \frac{3\sin A\sin B}{\cos A\cos B} = 1$$
$$\Rightarrow \frac{\cos A\cos B}{\sin A\sin B} = \frac{3}{1}$$
$$\Rightarrow \frac{\cos A\cos B+\sin A\sin B}{\cos A\cos B-\sin A\sin B} = \frac{3+1}{3-1} \quad \text{[Using: componendo-dividendo]}$$
$$\Rightarrow \frac{\cos(A-B)}{\cos(A+B)} = 2$$
$$\Rightarrow 2\cos(A+B) = \cos(A-B)$$

Type IV **ON USING THE FORMULAE:**

(i) $\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$

(ii) $\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$

(iii) $\tan(A\pm B) = \dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

EXAMPLE 12 *Prove that:*

(i) $\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$ [NCERT]

(ii) $\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$ [NCERT]

SOLUTION (i) We have,

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$$
$$= \left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)+\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)$$
$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\times\cos x = \sqrt{2}\cos x$$

(ii) We have,

$$\cos\left(\frac{3\pi}{4}+x\right)-\cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$$
$$= \left(\cos\frac{3\pi}{4}\cos x-\sin\frac{3\pi}{4}\sin x\right)+\left(\cos\frac{3\pi}{4}\cos x+\sin\frac{3\pi}{4}\sin x\right)$$
$$= \left(-\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)-\left(-\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$$
$$= -\frac{1}{\sqrt{2}}\sin x-\frac{1}{\sqrt{2}}\sin x = -\frac{2}{\sqrt{2}}\sin x = -\sqrt{2}\sin x$$

EXAMPLE 13 *Prove that:*

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x+\tan y}{\tan x-\tan y}$$ [NCERT]

SOLUTION We have,

$$\text{LHS} = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y-\cos x\sin y}$$
$$= \frac{\dfrac{\sin x\cos y+\cos x\sin y}{\cos x\cos y}}{\dfrac{\sin x\cos y-\cos x\sin y}{\cos x\cos y}} \quad \left[\text{Dividing the numerator and denominator by } \cos x\cos y\right]$$
$$= \frac{\tan x+\tan y}{\tan x-\tan y} = \text{RHS}$$

EXAMPLE 14 *Prove that:*

$$\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}+\frac{\sin(A-B)}{\cos A\cos B} = 0$$

SOLUTION We have,

$$\text{LHS} = \frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}+\frac{\sin(A-B)}{\cos A\cos B}$$
$$= \frac{\sin B\cos C-\cos B\sin C}{\cos B\cos C}+\frac{\sin C\cos A-\cos C\sin A}{\cos C\cos A}+\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}$$
$$= \frac{\sin B\cos C}{\cos B\cos C}-\frac{\cos B\sin C}{\cos B\cos C}+\frac{\sin C\cos A}{\cos C\cos A}-\frac{\cos C\sin A}{\cos C\cos A}+\frac{\sin A\cos B}{\cos A\cos B}-\frac{\cos A\sin B}{\cos A\cos B}$$
$$= \tan B-\tan C+\tan C-\tan A+\tan A-\tan B$$
$$= 0 = \text{RHS}$$

EXAMPLE 15 *If* $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha) = -\frac{3}{2}$, *prove that*

$$\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0$$

SOLUTION We have,

$$\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha) = -\frac{3}{2}$$
$$\Rightarrow 2\cos\alpha\cos\beta+2\sin\alpha\sin\beta+2\cos\beta\cos\gamma+2\sin\beta\sin\gamma+2\cos\gamma\cos\alpha+2\sin\gamma\sin\alpha = -3$$
$$\Rightarrow (2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha)+3 = 0$$
$$\Rightarrow (2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha+(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+(\cos^2\gamma+\sin^2\gamma) = 0$$
$$\Rightarrow (\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha)+(\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha) = 0$$

$\Rightarrow \quad (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow \quad \cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$

EXAMPLE 16 *If* $\sin B = 3\sin(2A+B)$, *prove that* $2\tan A + \tan(A+B) = 0$

SOLUTION We have,

$$\sin B = 3\sin(2A+B)$$

$$\Rightarrow \quad \frac{\sin(2A+B)}{\sin B} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{(A+B)+A\}}{\sin\{(A+B)-A\}} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{\sin\{(A+B)+A\} + \sin\{(A+B)-A\}}{\sin\{(A+B)+A\} - \sin\{(A+B)-A\}} = \frac{1+3}{1-3} \quad \text{[Using componendo-dividendo]}$$

$$\Rightarrow \quad \frac{\{\sin(A+B)\cos A + \cos(A+B)\sin A\} + \{\sin(A+B)\cos A - \cos(A+B)\sin A\}}{\{\sin(A+B)\cos A + \cos(A+B)\sin A\} - \{\sin(A+B)\cos A - \cos(A+B)\sin A\}} = \frac{1+3}{1-3}$$

$$\Rightarrow \quad \frac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = -2$$

$$\Rightarrow \quad \tan(A+B) = -2\tan A$$

$$\Rightarrow \quad 2\tan A + \tan(A+B) = 0$$

EXAMPLE 17 *If* $2\tan\beta + \cot\beta = \tan\alpha$, *prove that* $\cot\beta = 2\tan(\alpha-\beta)$.

SOLUTION We have,

$$2\tan(\alpha-\beta)$$

$$= 2\left\{\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right\}$$

$$= 2\left\{\frac{2\tan\beta + \cot\beta - \tan\beta}{1 + (2\tan\beta + \cot\beta)\tan\beta}\right\} \quad \text{[Using : } \tan\alpha = 2\tan\beta + \cot\beta\text{]}$$

$$= 2\left\{\frac{\tan\beta + \cot\beta}{1 + 2\tan^2\beta + 1}\right\}$$

$$= \frac{2(\tan\beta + \cot\beta)}{2 + 2\tan^2\beta} = \frac{2\left\{\tan\beta + \frac{1}{\tan\beta}\right\}}{2(1+\tan^2\beta)} = \frac{1}{\tan\beta} = \cot\beta$$

Hence, $\cot\beta = 2\tan(\alpha-\beta)$.

EXAMPLE 18 *If* $\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$, *prove that*

$$\cot\alpha\cot\beta\cot\gamma = \cot\delta$$

SOLUTION We have,

$$\cos(\alpha+\beta)\sin(\gamma+\delta) = \cos(\alpha-\beta)\sin(\gamma-\delta)$$

$$\Rightarrow \quad \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)}{\sin(\gamma-\delta)}$$

$$\Rightarrow \quad \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)} = \frac{\sin(\gamma+\delta)+\sin(\gamma-\delta)}{\sin(\gamma+\delta)-\sin(\gamma-\delta)} \quad \text{[Using componendo-dividendo]}$$

$$\Rightarrow \quad \frac{2\cos\alpha\cos\beta}{2\sin\alpha\sin\beta} = \frac{2\sin\gamma\cos\delta}{2\cos\gamma\sin\delta}$$

$$\Rightarrow \quad \cot\alpha\cot\beta = \tan\gamma\cot\delta$$

$$\Rightarrow \quad \cot\alpha\cot\beta\cot\gamma = \cot\delta$$

EXAMPLE 19 *Prove that:* $\frac{\sin(x+\theta)}{\sin(x+\phi)} = \cos(\theta-\phi) + \cot(x+\phi)\sin(\theta-\phi)$

SOLUTION We have,

$$\frac{\sin(x+\theta)}{\sin(x+\phi)} = \frac{\sin\{(x+\phi)+(\theta-\phi)\}}{\sin(x+\phi)}$$

$$= \frac{\sin(x+\phi)\cos(\theta-\phi) + \cos(x+\phi)\sin(\theta-\phi)}{\sin(x+\phi)}$$

$$= \cos(\theta-\phi) + \cot(x+\phi)\sin(\theta-\phi)$$

EXAMPLE 20 *Prove that:* $\frac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)} = \left(\frac{1+\tan x}{1-\tan x}\right)^2$ [NCERT]

SOLUTION We have,

$$\text{LHS} = \frac{\tan\left(\frac{\pi}{4}+x\right)}{\tan\left(\frac{\pi}{4}-x\right)} = \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} \times \frac{1 + \tan\frac{\pi}{4}\tan x}{\tan\frac{\pi}{4} - \tan x}$$

$$= \frac{1+\tan x}{1-\tan x} \times \frac{1+\tan x}{1-\tan x} = \left(\frac{1+\tan x}{1-\tan x}\right)^2 = \text{RHS}$$

EXAMPLE 21 *If* $\tan\beta = \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, *show that* $\tan(\alpha-\beta) = (1-n)\tan\alpha$

SOLUTION We have,

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$\Rightarrow \quad \tan(\alpha-\beta) = \frac{\frac{\sin\alpha}{\cos\alpha} - \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1 + \frac{\sin\alpha}{\cos\alpha} \times \frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}$$

$$\Rightarrow \quad \tan(\alpha-\beta) = \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha\cos^2\alpha}{\cos\alpha(1-n\sin^2\alpha) + n\sin^2\alpha\cos\alpha} \quad \text{[On taking LCM]}$$

$$\Rightarrow \quad \tan(\alpha-\beta) = \frac{\sin\alpha - n\sin^3\alpha - n\sin\alpha(1-\sin^2\alpha)}{\cos\alpha - n\sin^2\alpha\cos\alpha + n\sin^2\alpha\cos\alpha}$$

$$\Rightarrow \quad \tan(\alpha-\beta) = \frac{\sin\alpha - n\sin\alpha}{\cos\alpha} = \frac{\sin\alpha}{\cos\alpha}(1-n) = (1-n)\tan\alpha$$

EXAMPLE 22 *Prove that:*

(i) $\tan 3A\tan 2A\tan A = \tan 3A - \tan 2A - \tan A$ [NCERT]

(ii) $\cot A\cot 2A - \cot 2A\cot 3A - \cot 3A\cot A = 1$

SOLUTION (i) We have,

$$3A = 2A + A$$

$$\Rightarrow \quad \tan 3A = \tan(2A + A)$$

$$\Rightarrow \quad \tan 3A = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$\Rightarrow \quad \tan 3A\,(1 - \tan 2A \tan A) = \tan 2A + \tan A$$

$$\Rightarrow \quad \tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$$

$$\Rightarrow \quad \tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$$

(ii) Dividing both sides by $\tan A \tan 2A \tan 3A$, we get

$$\frac{\tan 3A \tan 2A \tan A}{\tan 3A \tan 2A \tan A} = \frac{\tan 3A - \tan 2A - \tan A}{\tan 3A \tan 2A \tan A}$$

$$\Rightarrow \quad 1 = \frac{1}{\tan 2A \tan A} - \frac{1}{\tan 3A \tan A} - \frac{1}{\tan 3A \tan 2A}$$

$$\Rightarrow \quad 1 = \cot A \cot 2A - \cot 3A \cot A - \cot 3A \cot 2A$$

EXAMPLE 23 *If α and β are acute angles such that* $\tan \alpha = \frac{m}{m+1}$ and $\tan \beta = \frac{1}{2m+1}$. *Prove that* $\alpha + \beta = \frac{\pi}{4}$

SOLUTION We have,

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{\frac{m}{m+1} + \frac{1}{2m+1}}{1 - \frac{m}{m+1} \times \frac{1}{2m+1}}$$

$$\Rightarrow \quad \tan(\alpha+\beta) = \frac{2m^2+m+m+1}{2m^2+3m+1-m} = \frac{2m^2+m+1}{2m^2+2m+1} = 1 = \tan\frac{\pi}{4}$$

$$\therefore \quad \alpha+\beta = \frac{\pi}{4}$$

EXAMPLE 24 *If* $A + B = \frac{\pi}{4}$, *prove that:*

(i) $(1+\tan A)(1+\tan B) = 2$ (ii) $(\cot A - 1)(\cot B - 1) = 2$

SOLUTION (i) We have,

$$A + B = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan(A+B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \quad \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$$

$$\Rightarrow \quad 1 + \tan A + \tan B + \tan A \tan B = 2$$

$$\Rightarrow \quad (1+\tan A) + \tan B\,(1+\tan A) = 2$$

$$\Rightarrow \quad (1+\tan A)(1+\tan B) = 2$$

(ii) We have,

$$A + B = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan(A+B) = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$$\Rightarrow \quad \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$$

$$\Rightarrow \quad \frac{\tan A + \tan B + \tan A \tan B}{\tan A \tan B} = \frac{1}{\tan A \tan B}$$

$$\Rightarrow \quad \cot B + \cot A + 1 = \cot A \cot B$$

$$\Rightarrow \quad \cot A \cot B - \cot A - \cot B = 1$$

$$\Rightarrow \quad \cot A \cot B - \cot A - \cot B + 1 = 2$$

$$\Rightarrow \quad \cot A(\cot B - 1) - (\cot B - 1) = 2$$

$$\Rightarrow \quad (\cot B - 1)(\cot A - 1) = 2$$

Type V **ON USING THE FORMULAE:**

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

EXAMPLE 25 *Prove that:* $\frac{\tan(A+B)}{\cot(A-B)} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$

SOLUTION We have,

$$\text{LHS} = \frac{\tan(A+B)}{\cot(A-B)} = \frac{\sin(A+B)\sin(A-B)}{\cos(A+B)\cos(A-B)} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B} = \text{RHS}$$

EXAMPLE 26 *Prove that:* $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$ **[NCERT]**

SOLUTION We have,

$$\sin^2 6x - \sin^2 4x = \sin(6x+4x)\sin(6x-4x) \qquad \left[\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)\right]$$

$$= \sin 10x \sin 2x$$

EXAMPLE 27 *Prove that:* $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$ **[NCERT]**

SOLUTION We have,

$$\cos^2 2x - \cos^2 6x$$

$$= (1 - \sin^2 2x) - (1 - \sin^2 6x)$$

$$= \sin^2 6x - \sin^2 2x = \sin(6x+2x)\sin(6x-2x) = \sin 8x \sin 4x$$

EXAMPLE 28 *Prove that*

$$\frac{\cos^2 33^\circ - \cos^2 57^\circ}{\sin^2\frac{21^\circ}{2} - \sin^2\frac{69^\circ}{2}} = -\sqrt{2}$$

SOLUTION We have,

$$\frac{\cos^2 33^\circ - \cos^2 57^\circ}{\sin^2\frac{21^\circ}{2} - \sin^2\frac{69^\circ}{2}} = \frac{(1-\sin^2 33^\circ) - (1-\sin^2 57^\circ)}{\sin^2\frac{21^\circ}{2} - \sin^2\frac{69^\circ}{2}} = \frac{\sin^2 57^\circ - \sin^2 33^\circ}{\sin^2\frac{21^\circ}{2} - \sin^2\frac{69^\circ}{2}}$$

$$= \frac{\sin(57°+33°)\sin(57°-33°)}{\sin\left(\frac{21°}{2}+\frac{69°}{2}\right)\sin\left(\frac{21°}{2}-\frac{69°}{2}\right)}$$

$$= \frac{\sin 90° \sin 24°}{\sin 45° \sin(-24°)} = \frac{\sin 24°}{-\frac{1}{\sqrt{2}}\sin 24°} = -\sqrt{2}$$

EXAMPLE 29 *Prove that:* $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)=\frac{1}{\sqrt{2}}\sin A$

SOLUTION We have,

$$\text{LHS} = \sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)$$

$$= \sin\left\{\left(\frac{\pi}{8}+\frac{A}{2}\right)+\left(\frac{\pi}{8}-\frac{A}{2}\right)\right\}\sin\left\{\left(\frac{\pi}{8}+\frac{A}{2}\right)-\left(\frac{\pi}{8}-\frac{A}{2}\right)\right\}$$

$$\left[\because \sin^2 A - \sin^2 B = \sin(A+B)-\sin(A-B)\right]$$

$$= \sin\frac{\pi}{4}\sin A = \frac{1}{\sqrt{2}}\sin A = \text{RHS}$$

EXAMPLE 30 *Prove that :*

$$\cos 2\alpha\cos 2\beta + \sin^2(\alpha-\beta)-\sin^2(\alpha+\beta) = \cos 2(\alpha+\beta)$$

SOLUTION We have,

$$\text{LHS} = \cos 2\alpha\cos 2\beta + \sin^2(\alpha-\beta)-\sin^2(\alpha+\beta) \quad \left[\begin{array}{r}\because \sin^2 A - \sin^2 B \\ = \sin(A+B)\sin(A-B)\end{array}\right]$$

$$= \cos 2\alpha\cos 2\beta + \sin(\alpha-\beta+\alpha+\beta)\sin(\alpha-\beta-\alpha-\beta)$$

$$= \cos 2\alpha\cos 2\beta + \sin 2\alpha\sin(-2\beta)$$

$$= \cos 2\alpha\cos 2\beta - \sin 2\alpha\sin 2\beta$$

$$= \cos(2\alpha+2\beta)$$

EXAMPLE 31 *Prove that:*

$$\sin^2 A = \cos^2(A-B)+\cos^2 B - 2\cos(A-B)\cos A\cos B.$$

SOLUTION We have,

$$\text{RHS} = \cos^2(A-B)+\cos^2 B - 2\cos(A-B)\cos A\cos B$$

$$= \cos^2 B + \cos^2(A-B) - 2\cos(A-B)\cos A\cos B$$

$$= \cos^2 B + \cos(A-B)\{\cos(A-B) - 2\cos A\cos B\}$$

$$= \cos^2 B + \cos(A-B)\{\cos A\cos B + \sin A\sin B - 2\cos A\cos B\}$$

$$= \cos^2 B + \cos(A-B)\{\sin A\sin B - \cos A\cos B\}$$

$$= \cos^2 B - \cos(A-B)(\cos A\cos B - \sin A\sin B)$$

$$= \cos^2 B - \cos(A-B)\cos(A+B) = \cos^2 B - (\cos^2 A - \sin^2 B)$$

$$= \cos^2 B + \sin^2 B - \cos^2 A = 1 - \cos^2 A = \sin^2 A = \text{LHS}$$

Type VI MISCELLANEOUS PROBLEMS

EXAMPLE 32 *If* $\cos(\alpha+\beta)=\frac{4}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$ *and* α, β *lie between 0 and* $\frac{\pi}{4}$, *prove that* $\tan 2\alpha = \frac{56}{33}$

SOLUTION Since α, β lie between 0 and $\pi/4$.

$-\pi/4 < \alpha-\beta < \pi/4$ and $0 < \alpha+\beta < \pi/2$.

$\therefore$ $\cos(\alpha-\beta)$ and $\sin(\alpha+\beta)$ are positive.

Now, $\sin(\alpha+\beta) = \sqrt{1-\cos^2(\alpha+\beta)} \Rightarrow \sin(\alpha+\beta) = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

and, $\cos(\alpha-\beta) = \sqrt{1-\sin^2(\alpha-\beta)} \Rightarrow \cos(\alpha-\beta) = \sqrt{1-\frac{25}{169}} = \frac{12}{13}$

$\therefore$ $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{3/5}{4/5} = \frac{3}{4}$ and, $\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{5}{12}$

$$\Rightarrow \tan 2\alpha = \tan[(\alpha+\beta)+(\alpha-\beta)] = \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)} = \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}$$

$$\Rightarrow \tan 2\alpha = \frac{56}{33}$$

EXAMPLE 33 *Prove that:* $\tan 70° = \tan 20° + 2\tan 50°$.

SOLUTION We have,

$$\tan A - \tan B = \tan(A-B)(1+\tan A\tan B)$$

$$\therefore \tan 70° - \tan 20° = \tan(70°-20°)(1+\tan 70°\tan 20°)$$

$$\Rightarrow \tan 70° - \tan 20° = \tan 50°\,(1+\tan 70°\cot 70°)$$

$$\Rightarrow \tan 70° - \tan 20° = 2\tan 50° \qquad [\because 1+\tan 70°\cot 70° = 2]$$

EXAMPLE 34 *If* $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$, *show that:* $(n+1)\sin 2\theta = (n-1)\sin 2\alpha$.

SOLUTION We have,

$$\tan(\alpha+\theta) = n\tan(\alpha-\theta)$$

$$\Rightarrow \frac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)} = \frac{n}{1}$$

$$\Rightarrow \frac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)} = \frac{n+1}{n-1} \quad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \frac{\sin(\alpha+\theta)\cos(\alpha-\theta)+\cos(\alpha+\theta)\sin(\alpha-\theta)}{\sin(\alpha+\theta)\cos(\alpha-\theta)-\cos(\alpha+\theta)\sin(\alpha-\theta)} = \frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin(\alpha+\theta+\alpha-\theta)}{\sin(\alpha+\theta-(\alpha-\theta))} = \frac{n+1}{n-1}$$

$$\Rightarrow \frac{\sin 2\alpha}{\sin 2\theta} = \frac{n+1}{n-1}$$

$$\Rightarrow (n+1)\sin 2\theta = (n-1)\sin 2\alpha$$

EXAMPLE 35 *If* $\tan A - \tan B = x$ *and* $\cot B - \cot A = y$, *prove that* $\cot(A-B) = \frac{1}{x}+\frac{1}{y}$

SOLUTION We have,

$$\tan A - \tan B = x \text{ and } \cot B - \cot A = y$$

Now,

$$\cot B - \cot A = y$$

$$\Rightarrow \quad \frac{\tan A - \tan B}{\tan A \tan B} = y$$

$$\Rightarrow \quad \frac{x}{\tan A \tan B} = y$$

$$\Rightarrow \quad \tan A \tan B = \frac{x}{y}$$

$$\therefore \quad \cot (A - B) = \frac{1}{\tan (A - B)}$$

$$= \frac{1 + \tan A \tan B}{\tan A - \tan B} = \frac{1 + \frac{x}{y}}{x} = \frac{x + y}{xy} = \frac{1}{x} + \frac{1}{y}$$

EXAMPLE 36 *If* $\tan \alpha = \frac{1}{\sqrt{x(x^2 + x + 1)}}$, $\tan \beta = \frac{\sqrt{x}}{\sqrt{x^2 + x + 1}}$ *and* $\tan \gamma = \sqrt{x^{-3} + x^{-2} + x^{-1}}$, *prove that* $\alpha + \beta = \gamma$.

SOLUTION We have,

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\Rightarrow \quad \tan (\alpha + \beta) = \frac{\frac{1}{\sqrt{x(x^2 + x + 1)}} + \frac{\sqrt{x}}{\sqrt{x^2 + x + 1}}}{1 - \frac{1}{\sqrt{x(x^2 + x + 1)}} \cdot \frac{\sqrt{x}}{\sqrt{x^2 + x + 1}}}$$

$$\Rightarrow \quad \tan (\alpha + \beta) = \frac{(x + 1)\sqrt{x(x^2 + x + 1)}}{x(x^2 + x + 1) - x}$$

$$\Rightarrow \quad \tan (\alpha + \beta) = \frac{(x + 1)\sqrt{x(x^2 + x + 1)}}{x^2 (x + 1)} = \sqrt{x^{-3} + x^{-2} + x^{-1}} = \tan \gamma$$

$$\therefore \quad \alpha + \beta = \gamma$$

EXAMPLE 37 *Prove that:* $\cot \theta \cot 2\theta + \cot 2\theta \cot 3\theta + 2 = \cot \theta (\cot \theta - \cot 3\theta)$

SOLUTION We know that

$$\cot A \cot B + 1 = \frac{\cos (A - B)}{\sin A \sin B}$$

$$\therefore \quad \text{LHS} = (\cot \theta \cot 2\theta + 1) + (\cot 2\theta \cot 3\theta + 1)$$

$$= \frac{\cos (2\theta - \theta)}{\sin \theta \sin 2\theta} + \frac{\cos (3\theta - 2\theta)}{\sin 2\theta \sin 3\theta}$$

$$= \frac{\cos \theta}{\sin \theta \sin 2\theta} + \frac{\cos \theta}{\sin 2\theta \sin 3\theta}$$

$$= \cos \theta \left\{ \frac{1}{\sin \theta \sin 2\theta} + \frac{1}{\sin 2\theta \sin 3\theta} \right\}$$

$$= \frac{\cos \theta}{\sin \theta} \left\{ \frac{\sin \theta}{\sin \theta \sin 2\theta} + \frac{\sin \theta}{\sin 2\theta \sin 3\theta} \right\}$$

$$= \cot \theta \left\{ \frac{\sin (2\theta - \theta)}{\sin \theta \sin 2\theta} + \frac{\sin (3\theta - 2\theta)}{\sin 2\theta \sin 3\theta} \right\}$$

$$= \cot \theta \{\cot \theta - \cot 2\theta + \cot 2\theta - \cot 3\theta\}$$

$$= \cot \theta (\cot \theta - \cot 3\theta)$$

$$= \text{RHS}$$

EXAMPLE 38 *If* $\tan (\pi \cos \theta) = \cot (\pi \sin \theta)$, *prove that* $\cos \left(\theta - \frac{\pi}{4} \right) = \pm \frac{1}{2\sqrt{2}}$

SOLUTION We have,

$$\tan (\pi \cos \theta) = \cot (\pi \sin \theta)$$

$$\Rightarrow \quad \frac{\sin (\pi \cos \theta)}{\cos (\pi \cos \theta)} = \frac{\cos (\pi \sin \theta)}{\sin (\pi \sin \theta)}$$

$$\Rightarrow \quad \sin (\pi \cos \theta) \sin (\pi \sin \theta) = \cos (\pi \sin \theta) \cos (\pi \cos \theta)$$

$$\Rightarrow \quad \cos (\pi \cos \theta) \cos (\pi \sin \theta) - \sin (\pi \cos \theta) \sin (\pi \sin \theta) = 0$$

$$\Rightarrow \quad \cos (\pi \cos \theta + \pi \sin \theta) = 0$$

$$\Rightarrow \quad \pi \cos \theta + \pi \sin \theta = \pm \frac{\pi}{2} \qquad \left[\because \cos \left(\pm \frac{\pi}{2} \right) = 0 \right]$$

$$\Rightarrow \quad \cos \theta + \sin \theta = \pm \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta = \pm \frac{1}{2\sqrt{2}} \qquad [\text{Multiplying both sides by } 1/\sqrt{2}]$$

$$\Rightarrow \quad \cos \theta \cos \frac{\pi}{4} + \sin \theta \sin \frac{\pi}{4} = \pm \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \quad \cos \left(\theta - \frac{\pi}{4} \right) = \pm \frac{1}{2\sqrt{2}}$$

EXAMPLE 39 *If* $a \tan \alpha + b \tan \beta = (a + b) \tan \left(\frac{\alpha + \beta}{2} \right)$, *where* $\alpha \neq \beta$, *prove that* $a \cos \beta = b \cos \alpha$.

SOLUTION We have,

$$a \tan \alpha + b \tan \beta = (a + b) \tan \left(\frac{\alpha + \beta}{2} \right)$$

$$\Rightarrow \quad a \left\{ \tan \alpha - \tan \left(\frac{\alpha + \beta}{2} \right) \right\} = b \left\{ \tan \left(\frac{\alpha + \beta}{2} \right) - \tan \beta \right\}$$

$$\Rightarrow \quad \frac{a \sin \left(\alpha - \frac{\alpha + \beta}{2} \right)}{\cos \alpha \cos \left(\frac{\alpha + \beta}{2} \right)} = \frac{b \sin \left(\frac{\alpha + \beta}{2} - \beta \right)}{\cos \left(\frac{\alpha + \beta}{2} \right) \cos \beta} \qquad \left[\because \tan A - \tan B = \frac{\sin (A - B)}{\cos A \cos B} \right]$$

$$\Rightarrow \quad \frac{a \sin \left(\frac{\alpha - \beta}{2} \right)}{\cos \alpha} = \frac{b \sin \left(\frac{\alpha - \beta}{2} \right)}{\cos \beta}$$

$$\Rightarrow \quad a \cos \beta = b \cos \alpha \qquad \left[\because \alpha \neq \beta \therefore \sin \left(\frac{\alpha - \beta}{2} \right) \neq 0 \right]$$

EXAMPLE 40 *If* $\sin \alpha + \sin \beta = a$ *and* $\cos \alpha + \cos \beta = b$, *show that*

(i) $\cos (\alpha + \beta) = \frac{b^2 - a^2}{b^2 + a^2}$ (ii) $\sin (\alpha + \beta) = \frac{2ab}{a^2 + b^2}$

REMARK *These relations are very useful to find the trigonometric ratios of the angles* $22\frac{1}{2}^\circ, 7\frac{1}{2}^\circ, 11\frac{1}{4}^\circ$ *etc.*

ILLUSTRATIVE EXAMPLES

Type I ON PROVING RESULTS AND IDENTITIES BASED UPON THE FOLLOWING FORMULAE: $\sin 2\theta = 2\sin\theta\cos\theta,\ 1+\cos 2\theta = 2\cos^2\theta,\ 1-\cos 2\theta = 2\sin^2\theta$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},\ 1+\cos\theta = 2\cos^2\frac{\theta}{2},\ 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

EXAMPLE 1 *Prove that:*

(i) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$ (ii) $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

(iii) $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \cot\theta$ (iv) $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\dfrac{\theta}{2}$

(v) $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \tan\left(\dfrac{\pi}{4}-\theta\right)$ (vi) $\dfrac{\cos\theta}{1+\sin\theta} = \tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

SOLUTION We have,

(i) $\text{LHS} = \dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta = \text{RHS}$

(ii) $\text{LHS} = \dfrac{\sin 2\theta}{1-\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta = \text{RHS}$

(iii) $\text{LHS} = \dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \dfrac{(1+\cos 2\theta)+\sin 2\theta}{(1-\cos 2\theta)+\sin 2\theta} = \dfrac{2\cos^2\theta+2\sin\theta\cos\theta}{2\sin^2\theta+2\sin\theta\cos\theta}$

$= \dfrac{2\cos\theta(\cos\theta+\sin\theta)}{2\sin\theta)(\cos\theta+\sin\theta)} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta = \text{RHS}$

(iv) $\text{LHS} = \dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \dfrac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$

$\Rightarrow \text{LHS} = \dfrac{2\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \dfrac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = \tan\dfrac{\theta}{2} = \text{RHS}$

(v) $\text{LHS} = \dfrac{\cos 2\theta}{1+\sin 2\theta}$

$\Rightarrow \text{LHS} = \dfrac{\sin\left(\frac{\pi}{2}-2\theta\right)}{1+\cos\left(\frac{\pi}{2}-2\theta\right)}$ $\left[\because \cos A = \sin\left(\frac{\pi}{2}-A\right), \sin A = \cos\left(\frac{\pi}{2}-A\right)\right]$

$\Rightarrow \text{LHS} = \dfrac{2\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4}-\theta\right)}$ $\left[\begin{array}{l}\because \sin A = 2\sin A/2\cos A/2\\ 1+\cos A = 2\cos^2\frac{A}{2}\end{array}\right]$

$\Rightarrow \text{LHS} = \tan\left(\dfrac{\pi}{4}-\theta\right) = \text{RHS}$

(vi) $\text{LHS} = \dfrac{\cos\theta}{1+\sin\theta} = \dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)}$

$\Rightarrow \text{LHS} = \dfrac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} = \tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right) = \text{RHS}$

EXAMPLE 2 *Show that:* $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$

SOLUTION We have,

$\text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$\Rightarrow \text{LHS} = \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}}$ $\left[\because 1+\cos 8\theta = 2\cos^2\frac{8\theta}{2}\right]$

$\Rightarrow \text{LHS} = \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4\theta)}}}$

$\Rightarrow \text{LHS} = \sqrt{2+\sqrt{2+2\cos 4\theta}}$

$\Rightarrow \text{LHS} = \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$

$\Rightarrow \text{LHS} = \sqrt{2+\sqrt{2(2\cos^2 2\theta)}}$ $[\because 1+\cos 4\theta = 2\cos^2 2\theta]$

$\Rightarrow \text{LHS} = \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)} = 2\cos\theta = \text{RHS}$

EXAMPLE 3 *Prove that:* $\dfrac{\sec 8\theta-1}{\sec 4\theta-1} = \dfrac{\tan 8\theta}{\tan 2\theta}$

SOLUTION We have,

$\text{LHS} = \dfrac{\sec 8\theta-1}{\sec 4\theta-1}$

$\Rightarrow \text{LHS} = \dfrac{\frac{1}{\cos 8\theta}-1}{\frac{1}{\cos 4\theta}-1} = \dfrac{1-\cos 8\theta}{\cos 8\theta}\cdot\dfrac{\cos 4\theta}{1-\cos 4\theta}$

$\Rightarrow \text{LHS} = \dfrac{2\sin^2 4\theta}{\cos 8\theta}\cdot\dfrac{\cos 4\theta}{2\sin^2 2\theta}$ $\left[\begin{array}{l}\because 1-\cos 8\theta = 2\sin^2\frac{8\theta}{2} = 2\sin^2 4\theta\\ \text{and, } 1-\cos 4\theta = 2\sin^2\frac{4\theta}{2} = 2\sin^2 2\theta\end{array}\right]$

$\Rightarrow \text{LHS} = \dfrac{(2\sin 4\theta\cos 4\theta)}{\cos 8\theta}\times\dfrac{\sin 4\theta}{2\sin^2 2\theta}$

$\Rightarrow \text{LHS} = \left(\dfrac{2\sin 4\theta\cos 4\theta}{\cos 8\theta}\right)\times\left(\dfrac{2\sin 2\theta\cos 2\theta}{2\sin^2 2\theta}\right)$

$\Rightarrow \text{LHS} = \left(\dfrac{\sin 2(4\theta)}{\cos 8\theta}\right)\times\left(\dfrac{\cos 2\theta}{\sin 2\theta}\right) = \left(\dfrac{\sin 8\theta}{\cos 8\theta}\right)\times\left(\dfrac{\cos 2\theta}{\sin 2\theta}\right) = \tan 8\theta\cot 2\theta$

$\Rightarrow \text{LHS} = \dfrac{\tan 8\theta}{\tan 2\theta} = \text{RHS}$

EXAMPLE 4 *Prove that:* $(\cos A+\cos B)^2+(\sin A-\sin B)^2 = 4\cos^2\left(\frac{A+B}{2}\right)$

SOLUTION We have,

$$\text{LHS} = (\cos A+\cos B)^2+(\sin A-\sin B)^2$$

$$\Rightarrow \text{LHS} = (\cos^2 A+\cos^2 B+2\cos A\cos B)+(\sin^2 A+\sin^2 B-2\sin A\sin B)$$

$$\Rightarrow \text{LHS} = (\cos^2 A+\sin^2 A)+(\cos^2 B+\sin^2 B)+2(\cos A\cos B-\sin A\sin B)$$

$$\Rightarrow \text{LHS} = 1+1+2\cos(A+B)$$

$$\Rightarrow \text{LHS} = 2+2\cos(A+B)$$

$$\Rightarrow \text{LHS} = 2\{1+\cos(A+B)\}$$

$$\Rightarrow \text{LHS} = 2\times 2\cos^2\left(\frac{A+B}{2}\right) \qquad \left[\because 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right]$$

$$\Rightarrow \text{LHS} = 4\cos^2\left(\frac{A+B}{2}\right) = \text{RHS}.$$

EXAMPLE 5 *Prove that:*

(i) $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$

(ii) $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=\frac{3}{2}$

SOLUTION We have,

(i) $$\frac{7\pi}{8} = \pi-\frac{\pi}{8} \text{ and } \frac{5\pi}{8} = \pi-\frac{3\pi}{8}$$

$$\Rightarrow \cos\frac{7\pi}{8} = -\cos\frac{\pi}{8} \text{ and } \cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$$

$$\Rightarrow \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8} \text{ and } \cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$$

$$\therefore \text{LHS} = 2\cos^4\frac{\pi}{8}+2\cos^4\frac{3\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\left[\left(\cos^2\frac{\pi}{8}\right)^2+\left(\cos^2\frac{3\pi}{8}\right)^2\right]$$

$$\Rightarrow \text{LHS} = 2\left[\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2+\left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2\right] \qquad \left[\because \frac{1+\cos 2\theta}{2} = \cos^2\theta\right]$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1+\cos\frac{\pi}{4}\right)^2+\left(1+\cos\frac{3\pi}{4}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{2}{4}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2+\left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right)+\left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{RHS}$$

(ii) We have,

$$\sin^4\frac{7\pi}{8} = \sin^4\left(\pi-\frac{\pi}{8}\right) = \sin^4\frac{\pi}{8} \text{ and, } \sin^4\frac{5\pi}{8} = \sin^4\left(\pi-\frac{3\pi}{8}\right) = \sin^4\frac{3\pi}{8}$$

$$\therefore \text{LHS} = \sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$$

$$\Rightarrow \text{LHS} = 2\left\{\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}\right\}$$

$$\Rightarrow \text{LHS} = 2\left\{\left(\sin^2\frac{\pi}{8}\right)^2+\left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$

$$\Rightarrow \text{LHS} = 2\left[\left\{\frac{1-\cos\pi/4}{2}\right\}^2+\left\{\frac{1-\cos 3\pi/4}{2}\right\}^2\right] \qquad \left[\because \frac{1-\cos 2\theta}{2} = \sin^2\theta\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)^2+\left(1-\cos\frac{3\pi}{4}\right)^2\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[\left(1-\frac{1}{\sqrt{2}}\right)^2+\left(1+\frac{1}{\sqrt{2}}\right)^2\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[\left(1+\frac{1}{2}-\sqrt{2}\right)+\left(1+\frac{1}{2}+\sqrt{2}\right)\right] = \frac{3}{2} = \text{RHS}$$

EXAMPLE 6 *Prove that:*

(i) $\cos^2 A+\cos^2\left(A+\frac{2\pi}{3}\right)+\cos^2\left(A-\frac{2\pi}{3}\right)=\frac{3}{2}$

(ii) $\cos^2 A+\cos^2\left(A+\frac{\pi}{3}\right)+\cos^2\left(A-\frac{\pi}{3}\right)=\frac{3}{2}$ **[NCERT]**

SOLUTION (i) We have,

$$\text{LHS} = \cos^2 A+\cos^2\left(A+\frac{2\pi}{3}\right)+\cos^2\left(A-\frac{2\pi}{3}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[2\cos^2 A+2\cos^2\left(A+\frac{2\pi}{3}\right)+2\cos^2\left(A-\frac{2\pi}{3}\right)\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[1+\cos 2A+\left\{1+\cos 2\left(A+\frac{2\pi}{3}\right)\right\}+\left\{1+\cos 2\left(A-\frac{2\pi}{3}\right)\right\}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[1+\cos 2A+1+\cos\left(2A+\frac{4\pi}{3}\right)+1+\cos\left(2A-\frac{4\pi}{3}\right)\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3+\cos 2A+\cos\left(2A+\frac{4\pi}{3}\right)+\cos\left(2A-\frac{4\pi}{3}\right)\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3+\cos 2A+2\cos 2A\cos\frac{4\pi}{3}\right] \qquad \left[\because \cos(A+B)+\cos(A-B) = 2\cos A\cos B\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left[3+\cos 2A+2(\cos 2A)\left(\frac{-1}{2}\right)\right] = \frac{1}{2}[3+\cos 2A-\cos 2A] = \frac{3}{2} = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \cos^2 A + \cos^2\left(A + \frac{\pi}{3}\right) + \cos^2\left(A - \frac{\pi}{3}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{2\cos^2 A + 2\cos^2\left(A + \frac{\pi}{3}\right) + 2\cos^2\left(A - \frac{\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{(1 + \cos 2A) + 1 + \cos\left(2A + \frac{2\pi}{3}\right) + 1 + \cos\left(2A - \frac{2\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{3 + \cos 2A + \cos\left(2A + \frac{2\pi}{3}\right) + \cos\left(2A - \frac{2\pi}{3}\right)\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{3 + \cos 2A + 2\cos 2A \cos\frac{2\pi}{3}\right\} \quad \left[\because \cos(A+B) + \cos(A-B) = 2\cos A \cos B\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\left\{3 + \cos 2A + 2(\cos 2A) \times -\frac{1}{2}\right\}$$

$$\Rightarrow \text{LHS} = \frac{1}{2}\{3 + \cos 2A - \cos 2A\} = \frac{3}{2} = \text{RHS}$$

EXAMPLE 7 *Prove that:* $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right) = \frac{1}{8}$

SOLUTION We have,

$$\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8} \text{ and } \cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$$

$$\therefore \text{LHS} = \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left[\left(1 - \cos\frac{\pi}{4}\right)\left(1 - \cos\frac{3\pi}{4}\right)\right] \quad \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$\Rightarrow \text{LHS} = \frac{1}{4}\left[\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} = \text{RHS}$$

EXAMPLE 8 *Prove that:* $\cos 4x = 1 - 8\sin^2 x \cos^2 x$. [NCERT]

SOLUTION We have,

$$\text{LHS} = \cos 4x = \cos 2(2x)$$

$$\Rightarrow \text{LHS} = 1 - 2\sin^2 2x = 1 - 2(\sin 2x)^2$$

$$\Rightarrow \text{LHS} = 1 - 2(2\sin x \cos x)^2 = 1 - 8\sin^2 x \cos^2 x = \text{RHS}$$

EXAMPLE 9 *Prove that*

$$\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \frac{1}{2}(\tan 27x - \tan x)$$

SOLUTION We have,

$$\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x}$$

$$= \frac{1}{2}\left\{\frac{2\sin x \cos x}{\cos 3x \cos x} + \frac{2\sin 3x \cos 3x}{\cos 9x \cos 3x} + \frac{2\sin 9x \cos 9x}{\cos 27x \cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(x + x)}{\cos 3x \cos x} + \frac{\sin(3x + 3x)}{\cos 9x \cos 3x} + \frac{\sin(9x + 9x)}{\cos 27x \cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin 2x}{\cos 3x \cos x} + \frac{\sin 6x}{\cos 9x \cos 3x} + \frac{\sin 18x}{\cos 27x \cos 9x}\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin(3x - x)}{\cos 3x \cos x} + \frac{\sin(9x - 3x)}{\cos 9x \cos 3x} + \frac{\sin(27x - 9x)}{\cos 27x \cos 9x}\right\}$$

$$= \frac{1}{2}\{(\tan 3x - \tan x) + (\tan 9x - \tan 3x) + (\tan 27x - \tan 9x)\}$$

$$= \frac{1}{2}(\tan 27x - \tan x)$$

EXAMPLE 10 *Prove that:*

(i) $\frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$ [NCERT]

(ii) $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \sin 4x$ [NCERT]

(iii) $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = 2\sin x$ [NCERT]

SOLUTION (i) We have,

$$\text{LHS} = \frac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x}$$

$$\Rightarrow \text{LHS} = \frac{(\sin 5x + \sin x) - 2\sin 3x}{\cos 5x - \cos x}$$

$$\Rightarrow \text{LHS} = \frac{2\sin\left(\frac{5x + x}{2}\right)\cos\left(\frac{5x - x}{2}\right) - 2\sin 3x}{-2\sin\left(\frac{5x + x}{2}\right)\sin\left(\frac{5x - x}{2}\right)}$$

$$\Rightarrow \text{LHS} = \frac{2\sin 3x \cos 2x - 2\sin 3x}{-2\sin 3x \sin 2x}$$

$$\Rightarrow \text{LHS} = -\frac{2\sin 3x(1 - \cos 2x)}{-2\sin 3x \sin 2x} = \frac{1 - \cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x \cos x} = \tan x = \text{RHS}$$

(ii) We have,

$$\text{LHS} = \sin 2x + 2\sin 4x + \sin 6x$$

$$\Rightarrow \text{LHS} = (\sin 6x + \sin 2x) + 2\sin 4x$$

$$\Rightarrow \text{LHS} = 2\sin\left(\frac{6x + 2x}{2}\right)\cos\left(\frac{6x - 2x}{2}\right) + 2\sin 4x$$

$$\Rightarrow \text{LHS} = 2\sin 4x \cos 2x + 2\sin 4x$$

$$\Rightarrow \text{LHS} = 2\sin 4x(\cos 2x + 1)$$

$$\Rightarrow \text{LHS} = 2\sin 4x \times 2\cos^2 x = 4\cos^2 x \sin 4x = \text{RHS}$$

(iii) We have,

$$LHS = \frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \frac{2\sin\left(\frac{x-3x}{2}\right)\cos\left(\frac{x+3x}{2}\right)}{-(\cos^2 x - \sin^2 x)}$$

$$\Rightarrow \quad LHS = \frac{2\sin(-x)\cos 2x}{-\cos 2x} = \frac{-2\sin x\cos 2x}{-\cos 2x} = 2\sin x = RHS$$

EXAMPLE 11 *Show that* $\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ = 4$

SOLUTION We have,

$$LHS = \sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$\Rightarrow \quad LHS = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ\cos 20^\circ}$$

$$\Rightarrow \quad LHS = \frac{2\left\{\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right\}}{\sin 20^\circ\cos 20^\circ}$$

$$\Rightarrow \quad LHS = \frac{2(\sin 60^\circ\cos 20^\circ - \cos 60^\circ\sin 20^\circ)}{\sin 20^\circ\cos 20^\circ}$$

$$\Rightarrow \quad LHS = \frac{2\sin(60^\circ - 20^\circ)}{\sin 20^\circ\cos 20^\circ}$$

$$\Rightarrow \quad LHS = \frac{2\sin 40^\circ}{\sin 20^\circ\cos 20^\circ} = \frac{4\sin 40^\circ}{2\sin 20^\circ\cos 20^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4 = RHS$$

EXAMPLE 12 *If* $\sin A = \frac{3}{5}$*, where* $0^\circ < A < 90^\circ$*, find the values of* $\sin 2A$*,* $\cos 2A$*,* $\tan 2A$ *and* $\sin 4A$

SOLUTION We have,

$$\sin A = \frac{3}{5}, \text{ where } 0^\circ < A < 90^\circ$$

$$\therefore \quad \cos^2 A = 1 - \sin^2 A$$

$$\Rightarrow \quad \cos A = +\sqrt{1 - \sin^2 A} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

$$\therefore \quad \tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$$

$$\sin 2A = 2\sin A\cos A = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\times\left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2\times\frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1 - \frac{9}{16}} = \frac{24}{7} \qquad \left[\because \tan A = \frac{3}{4}\right]$$

$$\sin 4A = 2\sin 2A\cos 2A = 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625} \qquad \left[\because \sin 2A = \frac{24}{25} \text{ and } \cos 2A = \frac{7}{25}\right]$$

EXAMPLE 13 *If* $\tan\alpha = \frac{1}{7}$*,* $\sin\beta = \frac{1}{\sqrt{10}}$*. Prove that* $\alpha + 2\beta = \frac{\pi}{4}$*, where* $0 < \alpha < \frac{\pi}{2}$ *and* $0 < \beta < \frac{\pi}{2}$

SOLUTION In order to prove that $\alpha + 2\beta = \frac{\pi}{4}$, it is sufficient to prove that $\tan(\alpha + 2\beta) = \tan\frac{\pi}{4} = 1$.

We have, $\sin\beta = \frac{1}{\sqrt{10}}$ and $0 < \beta < \frac{\pi}{2}$

$$\therefore \quad \cos\beta = \sqrt{1 - \sin^2\beta} = \sqrt{1 - \frac{1}{10}} = \frac{3}{\sqrt{10}}$$

and,
$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\frac{1}{\sqrt{10}}}{\frac{3}{\sqrt{10}}} = \frac{1}{3}$$

Now,
$$\tan 2\beta = \frac{2\tan\beta}{1 - \tan^2\beta}$$

$$\Rightarrow \quad \tan 2\beta = \frac{2\times\frac{1}{3}}{1 - \frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{2\times 9}{3\times 8} = \frac{3}{4}$$

Thus, we have

$$\tan\alpha = \frac{1}{7} \text{ and } \tan 2\beta = \frac{3}{4}$$

$$\therefore \quad \tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha\tan 2\beta}$$

$$\Rightarrow \quad \tan(\alpha + 2\beta) = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7}\times\frac{3}{4}} = \frac{4 + 21}{28 - 3} = 1$$

$$\Rightarrow \quad \alpha + 2\beta = \frac{\pi}{4}$$

EXAMPLE 14 *Prove that:*

$$\cos A\cos 2A\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

SOLUTION We have,

$$LHS = \cos A\cos 2A\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A$$

$$\Rightarrow \quad LHS = \frac{1}{2\sin A}\{(2\sin A\cos A)\cos 2A\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\}$$

$$\Rightarrow \quad LHS = \frac{1}{2\sin A}\{\sin 2A\cos 2A\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\}$$

$$\Rightarrow \quad LHS = \frac{1}{2^2\sin A}\{(2\sin 2A\cos 2A)\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\}$$

$$\Rightarrow \quad LHS = \frac{1}{2^2\sin A}\{\sin 2(2A)\cdot\cos 2^2 A\cos 2^3 A \ldots \cos 2^{n-1} A\}$$

$\Rightarrow \quad LHS = \frac{1}{2^3 \sin A} \left|(2 \sin 2^2 A \cos 2^2 A) \cos 2^3 A \ldots \cos 2^{n-1} A\right|$

$\Rightarrow \quad LHS = \frac{1}{2^3 \sin A} \left|\sin (2 \cdot 2^2 A) \cos 2^3 A \ldots \cos 2^{n-1} A\right|$

$\Rightarrow \quad LHS = \frac{1}{2^3 \sin A} \left|(\sin 2^3 A \cos 2^3 A \cos 2^4 A \ldots \cos 2^{n-1} A\right|$

$\Rightarrow \quad LHS = \ldots = \frac{1}{2^{n-1} \sin A} \left|\sin 2^{n-1} A \cos 2^{n-1} A\right|$

$\Rightarrow \quad LHS = \frac{1}{2^n \sin A} \left|2 \sin 2^{n-1} A \cos 2^{n-1} A\right|$

$\Rightarrow \quad LHS = \frac{1}{2^n \sin A} \sin (2 \times 2^{n-1} A) = \frac{1}{2^n \sin A} \sin 2^n A = RHS$

NOTE *The above relation can be treated as a standard formula.*

Type II **PROBLEMS BASED ON THE FORMULA:**

$$\cos A \cos 2A \cos 2^2 A \cos 2^3 A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}$$

EXAMPLE 15 *Prove that:* $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\frac{1}{8}$

SOLUTION Let $A = \frac{\pi}{7}$. Then,

$LHS = \cos A \cos 2A \cos 2^2 A$

$\Rightarrow \quad LHS = \frac{\sin 2^3 A}{2^3 \sin A}$

$\Rightarrow \quad LHS = \frac{\sin 8A}{8 \sin A} = \frac{\sin (7A + A)}{8 \sin A} = \frac{\sin (\pi + A)}{8 \sin A}$ $\quad [\because 7A = \pi]$

$\Rightarrow \quad LHS = \frac{-\sin A}{8 \sin A} = -\frac{1}{8}$

EXAMPLE 16 *Prove that:* $\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = \frac{1}{16}$

SOLUTION We have,

$LHS = \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \left(\pi - \frac{\pi}{15}\right)$

$\Rightarrow \quad LHS = \left(\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15}\right)\left(-\cos \frac{\pi}{15}\right)$

$\Rightarrow \quad LHS = -\cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15}$

$\Rightarrow \quad LHS = -\cos A \cos 2A \cos 2^2 A \cos 2^3 A$, where $A = \frac{\pi}{15}$

$\Rightarrow \quad LHS = -\frac{\sin 2^4 A}{2^4 \sin A} = -\frac{\sin 16A}{2^4 \sin A}$

$\Rightarrow \quad LHS = -\frac{\sin (15A + A)}{16 \sin A}$ $\quad [\because 15A = \pi]$

$\Rightarrow \quad LHS = \frac{-\sin (\pi + A)}{16 \sin A} = \frac{\sin A}{16 \sin A} = \frac{1}{16} = \frac{1}{16} = RHS$

EXAMPLE 17 *Prove that:* $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$

SOLUTION We have,

$LHS = \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$\Rightarrow \quad LHS = \frac{1}{2} (\cos 20^\circ \cos 40^\circ \cos 80^\circ)$

$\Rightarrow \quad LHS = \frac{1}{2} [\cos A \cos 2A \cos 2^2 A]$, where $A = 20^\circ$

$\Rightarrow \quad LHS = \frac{1}{2}\left(\frac{\sin 2^3 A}{2^3 \sin A}\right) = \frac{1}{2^4} \frac{\sin 8A}{\sin A}$

$\Rightarrow \quad LHS = \frac{1}{2^4} \frac{\sin 160^\circ}{\sin 20^\circ} = \frac{1}{2^4} \frac{\sin (180^\circ - 20^\circ)}{\sin 20^\circ} = \frac{1}{2^4} \frac{\sin 20^\circ}{\sin 20^\circ} = \frac{1}{2^4} = \frac{1}{16} = RHS$

EXAMPLE 18 *Prove that :*

$$\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14} = \frac{1}{64}$$

SOLUTION We know that

$A + B = \pi \Rightarrow \sin A = \sin (\pi - B) = \sin B$

$\because \quad \frac{\pi}{14} + \frac{13\pi}{14} = \pi, \ \frac{3\pi}{14} + \frac{11\pi}{14} = \pi, \ \frac{5\pi}{14} + \frac{9\pi}{14} = \pi$

$\therefore \quad \sin \frac{\pi}{14} = \sin \frac{13\pi}{14}, \ \sin \frac{3\pi}{14} = \sin \frac{11\pi}{14}, \ \sin \frac{5\pi}{14} = \sin \frac{9\pi}{14}$

Now,

$LHS = \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}$

$\Rightarrow \quad LHS = \left\{\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}\right\}^2 \times \sin \frac{7\pi}{14}$

$\Rightarrow \quad LHS = \left\{\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}\right\}^2$

$\Rightarrow \quad LHS = \left\{\cos \left(\frac{\pi}{2} - \frac{\pi}{14}\right) \cos \left(\frac{\pi}{2} - \frac{3\pi}{14}\right) \cos \left(\frac{\pi}{2} - \frac{5\pi}{14}\right)\right\}^2$

$\Rightarrow \quad LHS = \left\{\cos \frac{6\pi}{14} \cos \frac{4\pi}{14} \cos \frac{2\pi}{14}\right\}^2$

$\Rightarrow \quad LHS = \left[\cos \left(\pi - \frac{8\pi}{14}\right) \cos \frac{4\pi}{14} \cos \frac{2\pi}{14}\right]^2$

$\Rightarrow \quad LHS = \left\{-\cos \frac{8\pi}{14} \cos \frac{4\pi}{14} \cos \frac{2\pi}{14}\right\}^2 = \left\{-\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}\right\}^2$

$\Rightarrow \quad LHS = \left[-\left(-\frac{1}{8}\right)\right]^2 = \frac{1}{64}$ [See Ex. 15]

EXAMPLE 19 *Find the value of* $\sin \frac{\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18}$

SOLUTION We have,

$\sin \frac{\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18}$

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13.11.1 PROPERTIES OF MODULUS

THEOREM *If* $z, z_1, z_2 \in C$, *then*

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{Im}(z) = 0$

(ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \le \text{Re}(z) \le |z|, \; -|z| \le \text{Im}(z) \le |z|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1|\,|z_2|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|};\; z_2 \ne 0$

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \bar{z}_2)$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1 \bar{z}_2)$

(ix) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(x) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$, *where* $a, b \in R$.

PROOF Let $z = a + ib$. Then,

(i) $\quad |z| = 0$

$\Leftrightarrow \quad \sqrt{a^2 + b^2} = 0$

$\Leftrightarrow \quad a^2 + b^2 = 0$

$\Leftrightarrow \quad a = 0$ and $b = 0$

$\Leftrightarrow \quad \text{Re}(z) = \text{Im}(z) = 0$

(ii) Let $z = a + ib$. Then, $\bar{z} = a - ib$ and $-z = -a - ib$.

$\therefore \quad |z| = \sqrt{a^2 + b^2}, \; |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$

and, $\quad |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Hence, $\quad |z| = |\bar{z}| = |-z|$

(iii) Let $z = a + ib$. Then, $|z| = \sqrt{a^2 + b^2}$

Now, $\quad -\sqrt{a^2 + b^2} \le a \le \sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2} \le b \le \sqrt{a^2 + b^2}$

$\Rightarrow \quad -|z| \le \text{Re}(z) \le |z|$ and $-|z| \le \text{Im}(z) \le |z|$

(iv) Let $z = a + ib$. Then, $\bar{z} = a - ib$.

$$\therefore \quad z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \left|\sqrt{a^2 + b^2}\right|^2 = |z|^2$$

(v) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

$$\Rightarrow \quad |z_1 z_2| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2}$$

$$\Rightarrow \quad |z_1 z_2| = \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2}$$

$$\Rightarrow \quad |z_1 z_2| = \sqrt{a_1^2 (a_2^2 + b_2^2) + b_1^2 (a_2^2 + b_2^2)} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

$$\Rightarrow \quad |z_1 z_2| = \sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2} = |z_1|\,|z_2|$$

(vi) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = (a_1 + ib_1)\left(\frac{1}{a_2 + ib_2}\right) = (a_1 + ib_1)\left(\frac{a_2}{a_2^2 + b_2^2} + i\frac{(-b_2)}{a_2^2 + b_2^2}\right)$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right)^2 + \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)^2}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1 a_2 + b_1 b_2)^2 + (a_2 b_1 - a_1 b_2)^2}{(a_2^2 + b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{a_1^2 a_2^2 + b_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2}{(a_2^2 + b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}{(a_2^2 + b_2^2)^2}} = \sqrt{\frac{a_1^2 + b_1^2}{a_2^2 + b_2^2}} = \frac{\sqrt{a_1^2 + b_1^2}}{\sqrt{a_2^2 + b_2^2}} = \frac{|z_1|}{|z_2|}$$

(vii) We have,

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$ $\quad [\because z\bar{z} = |z|^2]$

$\Rightarrow \quad |z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$ $\quad [\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2]$

$\Rightarrow \quad |z_1 + z_2|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + z_2\bar{z}_1$ $\quad$ [By distributivity of mult.]

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)}$

$\quad [\because \overline{(z_1\bar{z}_2)} = \bar{z}_1(\bar{\bar{z}}_2) = \bar{z}_1 z_2 = z_2\bar{z}_1]$

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$ $\quad [\because z + \bar{z} = 2\,\text{Re}(z)]$

(viii) We have,

$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)}$ $\quad [\because z\bar{z} = |z|^2]$

$\Rightarrow \quad |z_1 - z_2|^2 = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ $\quad [\because \overline{z_1 + z_2} = \bar{z}_1 - \bar{z}_2]$

$\Rightarrow \quad |z_1 - z_2|^2 = z_1\bar{z}_1 + z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1$ $\quad$ [By distributivity of multiplication]

$\Rightarrow \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \overline{(z_1\bar{z}_2)}$ $\quad [\because \overline{(z_1\bar{z}_2)} = \bar{z}_1(\bar{\bar{z}}_2) = \bar{z}_1 z_2]$

$\Rightarrow \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$

(ix) Using (vii) and (viii), we get

$|z_1 + z_2|^2 + |z_1 - z_2|^2$

$= |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2) + |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$

$= 2(|z_1|^2 + |z_2|^2)$

(x) We have,

$$|az_1 - bz_2|^2 = (az_1 - bz_2)\overline{(az_1 - bz_2)}$$
$$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2)$$
$$= a^2 z_1\bar{z}_1 - az_1 b\bar{z}_2 - bz_2 a\bar{z}_1 + b^2 z_2\bar{z}_2$$
$$= a^2|z_1|^2 - ab(z_1\bar{z}_2 + \bar{z}_1 z_2) + b^2|z_2|^2$$
$$= a^2|z_1|^2 - ab(z_1\bar{z}_2 + \overline{(z_1\bar{z}_2)}) + b^2|z_2|^2$$

$$= a^2 |z_1|^2 - ab\,(2\,\mathrm{Re}\,(z_1 \bar{z}_2)) + b^2 |z_2|^2 \qquad \left[\because z_1 \bar{z}_2 + \overline{(z_1 \bar{z}_2)} = 2\,\mathrm{Re}\,(z_1 \bar{z}_2)\right]$$

$$= a^2 |z_1|^2 - 2ab\,\mathrm{Re}\,(z_1 \bar{z}_2) + b^2 |z_2|^2$$

Similarly, we have

$$|b z_1 + a z_2|^2 = b^2 |z_1|^2 + a^2 |z_2|^2 + 2ab\,\mathrm{Re}\,(z_1 \bar{z}_2)$$

$$\therefore \quad |a z_1 - b z_2|^2 + |b z_1 + a z_2|^2$$

$$= a^2 |z_1|^2 - 2ab\,\mathrm{Re}\,(z_1 \bar{z}_2) + b^2 |z_2|^2 + b^2 |z_1|^2 + a^2 |z_2|^2 + 2ab\,\mathrm{Re}\,(z_1 \bar{z}_2)$$

$$= |z_1|^2 (a^2 + b^2) + |z_2|^2 (b^2 + a^2)$$

$$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

13.12 RECIPROCAL OF A COMPLEX NUMBER

Let $z = a + ib$ be a non-zero complex number. Then,

$$\frac{1}{z} = \frac{1}{a + ib} = \frac{1}{a + ib} \times \frac{a - ib}{a - ib} \qquad \left[\begin{array}{l}\text{Multiplying numerator and denominator}\\ \text{by conjugate of denominator}\end{array}\right]$$

$$\Rightarrow \quad \frac{1}{z} = \frac{a - ib}{a^2 - i^2 b^2} = \frac{a - ib}{a^2 + b^2}$$

$$\Rightarrow \quad \frac{1}{z} = \frac{a}{a^2 + b^2} + \frac{i(-b)}{a^2 + b^2}$$

Clearly, $\frac{1}{z}$ is equal to the multiplicative inverse of z.

Also, $\quad \frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$

Thus, the multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by

$$\frac{\mathrm{Re}\,(z)}{|z|^2} + i\,\frac{(-\mathrm{Im}\,(z))}{|z|^2} = \frac{\bar{z}}{|z|^2}$$

ILLUSTRATIVE EXAMPLES

Type I **EXPRESSING A COMPLEX NUMBER IN THE STANDARD FORM $a + ib$**

In order to express a complex number in the standard form, we may follow the following algorithm.

ALGORITHM

STEP I *Write the complex number in the form $\frac{a + ib}{c + id}$ by using fundamental operations of addition, subtraction and multiplication.*

STEP II *Multiply the numerator and denominator by the conjugate of the denominator.*

EXAMPLE 1 *Express the following in the form $a + ib$:*

(i) $(-5i)\left(\frac{1}{8}i\right)$ [NCERT] (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$ [NCERT] (iii) $(5i)\left(-\frac{3}{5}i\right)$ [NCERT]

(iv) $i^9 + i^{19}$ (v) i^{-39} [NCERT] (vi) $(1 - i)^4$ [NCERT]

SOLUTION We have,

(i) $(-5i)\left(\frac{1}{8}i\right) = -\frac{5}{8}i^2 = -\frac{5}{8} \times -1 = \frac{5}{8} = \frac{5}{8} + 0\,i$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3 = -2i^2 \times -\frac{1}{512}i^3 = \frac{1}{256} \times i^2 \times i^3 = \frac{1}{256}i^5 = \frac{i}{256} = 0 + \frac{1}{256}i$

(iii) $(5i)\left(-\frac{3}{5}i\right) = -3i^2 = -3 \times -1 = 3 = 3 + 0\,i$

(iv) $i^9 + i^{19} = (i^4)^2 i + (i^4)^4 i^3 = i + i^3 = i - i = 0 = 0 + 0\,i$

(v) $i^{-39} = (i^4)^{-10} i = i = 0 + 1\,i$

(vi) $(1 - i)^4 = \{(1 - i)^2\}^2 = (1 - 2i + i^2)^2 = (1 - 2i - 1)^2 = (-2i)^2 = 4i^2 = -4 = -4 + 0i$

EXAMPLE 2 *Express each of the following in the form $a + ib$:*

(i) $3(7 + 7i) + i(7 + 7i)$ [NCERT] (ii) $(1 - i) - (-1 + 6i)$ [NCERT]

(iii) $\left(\frac{1}{5} + \frac{2}{5}i\right) - \left(4 + \frac{5}{2}i\right)$ [NCERT] (iv) $\left[\left(\frac{1}{3} + \frac{7}{3}i\right) + \left(4 + \frac{1}{3}i\right)\right] - \left(-\frac{4}{3} + i\right)$ [NCERT]

SOLUTION We have,

(i) $3(7 + 7i) + i(7 + 7i) = 21 + 21\,i + 7i + 7i^2 = 21 + 21\,i + 7i - 7 = 14 + 28\,i$

(ii) $(1 - i) - (-1 + 6i) = 1 - i + 1 - 6i = 2 - 7i$

(iii) $\left(\frac{1}{5} + \frac{2}{5}i\right) - \left(4 + \frac{5}{2}i\right) = \left(\frac{1}{5} - 4\right) + \frac{2i}{5} - \frac{5i}{2} = -\frac{19}{5} - \frac{21}{10}i$

(iv) $\left[\left(\frac{1}{3} + \frac{7}{3}i\right) + \left(4 + \frac{1}{3}i\right)\right] - \left(-\frac{4}{3} + i\right) = \left[\left(\frac{1}{3} + 4\right) + i\left(\frac{7}{3} + \frac{1}{3}\right)\right] - \left(-\frac{4}{3} + i\right)$

$$= \left(\frac{13}{3} + \frac{8}{3}i\right) + \frac{4}{3} - i$$

$$= \left(\frac{13}{3} + \frac{4}{3}\right) + \left(\frac{8}{3} - 1\right)i = \frac{17}{3} + \frac{5}{3}i$$

EXAMPLE 3 *Express each of the following in the form $a + ib$:*

(i) $\left(\frac{1}{3} + 3i\right)^3$ [NCERT] (ii) $\left(-2 - \frac{1}{3}i\right)^3$ [NCERT]

(iii) $(5 - 3i)^3$ [NCERT] (iv) $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$

SOLUTION We have,

(i) $\left(\frac{1}{3} + 3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3 \times \frac{1}{3} \times 3i\left(\frac{1}{3} + 3i\right)$

$$= \frac{1}{27} + 27i^3 + 3i\left(\frac{1}{3} + 3i\right)$$

$$= \frac{1}{27} + 27i^3 + i + 9i^2 = \frac{1}{27} - 27i + i - 9 = -\frac{242}{27} - 26i$$

(ii) $\left(-2 - \frac{1}{3}i\right)^3 = (-2)^3 + \left(-\frac{1}{3}i\right)^3 + 3 \times -2 \times -\frac{1}{3}i\left(-2 - \frac{1}{3}i\right)$

$$= -8 - \frac{1}{27} i^3 + 2i\left(-2 - \frac{1}{3} i\right)$$

$$= -8 + \frac{1}{27} i - 4i - \frac{2}{3} i^2 = -8 + \frac{1}{27} i - 4i + \frac{2}{3} = -\frac{22}{3} - \frac{107}{27} i$$

(iii) $(5-3i)^3 = 5^3 + (-3i)^3 + 3 \times 25 \times -3i + 3 \times 5 \times (-3i)^2$

$$= 125 + 27i - 225i - 135 = -10 - 198i$$

(v) $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i) = (-\sqrt{3} + i\sqrt{2})(2\sqrt{3} - i)$

$$= -6 + \sqrt{3}i + 2\sqrt{6}i - \sqrt{2}i^2$$

$$= -6 + (\sqrt{3} + 2\sqrt{6})i + \sqrt{2} = (\sqrt{2} - 6) + (\sqrt{3} + 2\sqrt{6})i$$

EXAMPLE 4 *Express each one of the following in the standard form $a + ib$.*

(i) $\frac{1}{3-4i}$ (ii) $\frac{5+4i}{4+5i}$ (iii) $\frac{(1+i)^2}{3-i}$

(iv) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ (v) $\frac{1}{-2+\sqrt{-3}}$ (vi) $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

(vii) $\frac{1}{1-\cos\theta + 2i\sin\theta}$ (viii) $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$ [NCERT]

SOLUTION We have,

(i) $\frac{1}{3-4i} = \frac{1}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i}{9-16i^2} = \frac{3+4i}{9+16} = \frac{3}{25} + \frac{4}{25} i$

(ii) $\frac{5+4i}{4+5i} = \frac{5+4i}{4+5i} \times \frac{4-5i}{4-5i} = \frac{(20+20)+i(16-25)}{16-25i^2} = \frac{40-9i}{41} = \frac{40}{41} - \frac{9}{41} i$

(iii) $\frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} = \frac{2i}{3-i} = \frac{2i}{3-i} \cdot \frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} = \frac{-2+6i}{10} = -\frac{1}{5} + \frac{3}{5} i$

(iv) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{(6+6)+i(-4+9)}{(2+2)+i(4-1)} = \frac{12+5i}{4+3i}$

$$= \frac{12+5i}{4+3i} \cdot \frac{4-3i}{4-3i} = \frac{(48+15)+i(-36+20)}{16-9i^2} = \frac{63}{25} - \frac{16}{25} i$$

(v) $\frac{1}{-2+\sqrt{-3}} = \frac{1}{-2+i\sqrt{3}} = \frac{1}{-2+i\sqrt{3}} \times \frac{-2-i\sqrt{3}}{-2-i\sqrt{3}} = \frac{-2-i\sqrt{3}}{4-3i^2} = \frac{-2}{7} - \frac{\sqrt{3}}{7} i$

(vi) $\left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

$$= \frac{1+i+3-6i}{(1+2)+i(-2+1)} \times \frac{3+4i}{2-4i}$$

$$= \frac{4-5i}{3-i} \cdot \frac{3+4i}{2-4i} = \frac{(12+20)+i(16-15)}{(6-4)+i(-2-12)} = \frac{32+i}{2-14i} = \frac{32+i}{2-14i} \cdot \frac{2+14i}{2+14i}$$

$$= \frac{(64-14)+i(2+448)}{4-196i^2} = \frac{50+450i}{200} = \frac{1}{4} + \frac{9}{4} i$$

(vii) $\frac{1}{1-\cos\theta+2i\sin\theta} = \frac{1}{1-\cos\theta+2i\sin\theta} \times \frac{1-\cos\theta-2i\sin\theta}{1-\cos\theta-2i\sin\theta}$

$$= \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2-4i^2\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta}$$

$$= \frac{1-\cos\theta-2i\sin\theta}{1-2\cos\theta+\cos^2\theta+4\sin^2\theta} = \frac{1-\cos\theta-2i\sin\theta}{2-2\cos\theta+3\sin^2\theta}$$

$$= \left(\frac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}\right) + i\left(\frac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}\right)$$

(viii) $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})} = \frac{(9-\sqrt{5}\times-\sqrt{5})+i(3\times-\sqrt{5}+3\sqrt{5})}{\sqrt{3}+\sqrt{2}i-\sqrt{3}+\sqrt{2}i}$

$$= \frac{(9+5)+i\times 0}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}i} = \frac{7}{\sqrt{2}i} = \frac{-7}{\sqrt{2}} i = 0 - \frac{7}{\sqrt{2}} i$$

EXAMPLE 5 *Prove that the following complex numbers are purely real:*

(i) $\left(\frac{2+3i}{3+4i}\right)\left(\frac{2-3i}{3-4i}\right)$ (ii) $\left(\frac{3+2i}{2-3i}\right) + \left(\frac{3-2i}{2+3i}\right)$

SOLUTION We have,

(i) $\left(\frac{2+3i}{3+4i}\right)\left(\frac{2-3i}{3-4i}\right) = \frac{(2+3i)(2-3i)}{(3+4i)(3-4i)} = \frac{4-9i^2}{9-16i^2} = \frac{13}{25}$, which is purely real.

(ii) $\left(\frac{3+2i}{2-3i}\right) + \left(\frac{3-2i}{2+3i}\right) = \frac{3+2i}{2-3i} \times \frac{2+3i}{2+3i} + \frac{3-2i}{2+3i} \times \frac{2-3i}{2-3i}$

$$= \frac{(3+2i)(2+3i)}{4-9i^2} + \frac{(3-2i)(2-3i)}{4-9i^2}$$

$$= \frac{13i}{13} - \frac{13i}{13} = 0$$, which is purely real.

EXAMPLE 6 *Express $(1-2i)^{-3}$ in the standard form $a + ib$.*

SOLUTION We have,

$$(1-2i)^{-3} = \frac{1}{(1-2i)^3} = \frac{1}{1-8i^3-6i+12i^2}$$

$$\Rightarrow (1-2i)^{-3} = \frac{1}{1+8i-6i-12} = \frac{1}{-11+2i}$$

$$\Rightarrow (1-2i)^{-3} = \frac{1}{-11+2i} \times \frac{-11-2i}{-11-2i}$$

$$\Rightarrow (1-2i)^{-3} = \frac{-11-2i}{(-11)^2-(2i)^2} = \frac{-11-2i}{125} = \frac{-11}{125} - \frac{2}{125} i$$

EXAMPLE 7 *Perform the indicated operation and find the result in the form $a + ib$*

(i) $\frac{2-\sqrt{-25}}{1-\sqrt{-16}}$ (ii) $\frac{3-\sqrt{-16}}{1-\sqrt{-9}}$

SOLUTION We have,

(i) $\frac{2-\sqrt{-25}}{1-\sqrt{-16}} = \frac{2-5i}{1-4i} = \frac{2-5i}{1-4i} \times \frac{1+4i}{1+4i}$

$$\Rightarrow \frac{2-\sqrt{-25}}{1-\sqrt{-16}} = \frac{(2+20)+i(8-5)}{1-16i^2} = \frac{22+3i}{17} = \frac{22}{17} + \frac{3}{17} i$$

(ii) $\frac{3-\sqrt{-16}}{1-\sqrt{-9}} = \frac{3-4i}{1-3i} = \frac{3-4i}{1-3i} \times \frac{1+3i}{1+3i}$

$$\Rightarrow \frac{3-\sqrt{-16}}{1-\sqrt{-9}} = \frac{(3+12)+i(-4+9)}{1-9i^2} = \frac{15}{10} + \frac{5}{10} i = \frac{3}{2} + \frac{1}{2} i$$

EXAMPLE 8 *If z_1, z_2 are $1-i, -2+4i$, respectively, find* $\text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$.

SOLUTION We have,

$$\frac{z_1 z_2}{\bar{z}_1} = \frac{(1-i)(-2+4i)}{\overline{(1-i)}} = \frac{(-2+4)+i(2+4)}{1+i}$$

$$\Rightarrow \quad \frac{z_1 z_2}{\bar{z}_1} = \frac{2+6i}{1+i} = \frac{2+6i}{1+i} \times \frac{1-i}{1-i} = \frac{(2+6)+i(6-2)}{1+1} = 4+2i$$

$$\Rightarrow \quad \text{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = 2$$

Type II **ON EQUALITY OF COMPLEX NUMBERS**

Recall that two complex numbers z_1 and z_2 are equal iff

$$\text{Re}(z_1) = \text{Re}(z_2) \text{ and } \text{Im}(z_1) = \text{Im}(z_2).$$

EXAMPLE 9 *Find the real values of x and y, if*

(i) $(3x-7)+2iy = -5y+(5+x)i$ (ii) $(1-i)x+(1+i)y = 1-3i$

(iii) $(x+iy)(2-3i) = 4+i$ (iv) $\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$

SOLUTION We have,

(i) $(3x-7)+2iy = -5y+(5+x)i$

$\Rightarrow \quad 3x-7 = -5y$ and $2y = 5+x$

$\Rightarrow \quad 3x+5y = 7$ and $x-2y = -5$

$\Rightarrow \quad x = -1, y = 2.$

(ii) $(1-i)x+(1+i)y = 1-3i$

$\Rightarrow \quad (x+y)+i(-x+y) = 1-3i$

$\Rightarrow \quad x+y = 1$ and $-x+y = -3$

$\Rightarrow \quad x = 2, y = -1$

(iii) $(x+iy)(2-3i) = 4+i$

$\Rightarrow \quad (2x+3y)+i(-3x+2y) = 4+i$

$\Rightarrow \quad 2x+3y = 4$ and $-3x+2y = 1$

$\Rightarrow \quad x = \frac{5}{13}, y = \frac{14}{13}$

(iv) $\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$

$$\Rightarrow \quad \frac{(x-1)(3-i)+(y-1)(3+i)}{(3+i)(3-i)} = i$$

$$\Rightarrow \quad \frac{(3x+3y-6)+i(y-x)}{9-i^2} = i$$

$$\Rightarrow \quad \left(\frac{3x+3y-6}{10}\right) + i\left(\frac{y-x}{10}\right) = 0+i$$

$$\Rightarrow \quad \frac{3x+3y-6}{10} = 0 \text{ and } \frac{y-x}{10} = 1$$

$\Rightarrow \quad x+y-2 = 0$ and $y-x = 10$

$\Rightarrow \quad x = -4, y = 6.$

EXAMPLE 10 *Find real values of x and y for which the following equalities hold:*

(i) $(1+i)y^2+(6+i) = (2+i)x$ (ii) $(x^4+2xi)-(3x^2+iy) = (3-5i)+(1+2iy)$

SOLUTION We have,

(i) $(1+i)y^2+(6+i) = (2+i)x$

$\Rightarrow \quad (y^2+6)+i(y^2+1) = 2x+ix$

$\Rightarrow \quad y^2+6 = 2x$ ...(i)

and $\quad y^2+1 = x$ ...(ii)

From (i) and (ii), we get

$$y^2+6 = 2(y^2+1) \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Now, $\quad x = y^2+1 \Rightarrow x = 5$ when $y = \pm 2$

Thus, $\quad x = 5$ and $y = 2$ or, $x = 5$ and $y = -2$

(ii) $(x^4+2xi)-(3x^2+iy) = (3-5i)+(1+2iy)$

$\Rightarrow \quad (x^4-3x^2)+i(2x-y) = 4+i(2y-5)$

$\Rightarrow \quad x^4-3x^2 = 4$ and $2x-y = 2y-5$

$\Rightarrow \quad x^4-3x^2-4 = 0, \; 2x-3y+5 = 0$

Now,

$$x^4-3x^2-4 = 0$$

$\Rightarrow \quad (x^2-4)(x^2+1) = 0$

$\Rightarrow \quad x^2-4 = 0 \Rightarrow x = \pm 2$ $\quad [\because x^2+1 \neq 0$ for any real $x]$

Putting $x = \pm 2$ in $2x-3y+5 = 0$, we get

$y = 3$ when $x = 2$ and $y = 1/3$ when $x = -2$

Thus, $\quad x = 2$ and $y = 3$ or, $x = -2$ and $y = 1/3$.

EXAMPLE 11 *If $a+ib = \frac{c+i}{c-i}$, where c is real, prove that:*

$$a^2+b^2 = 1 \text{ and } \frac{b}{a} = \frac{2c}{c^2-1}$$

SOLUTION We have,

$$a+ib = \frac{c+i}{c-i}$$

$$\Rightarrow \quad a+ib = \frac{(c+i)(c+i)}{(c-i)(c+i)}$$

$$\Rightarrow \quad a+ib = \frac{(c+i)^2}{c^2-i^2}$$

$$\Rightarrow \quad a+ib = \frac{c^2+2ic+i^2}{c^2-i^2}$$

$$\Rightarrow \quad a+ib = \frac{c^2-1}{c^2+1} + \frac{i2c}{c^2+1}$$

$$\Rightarrow \quad a = \frac{c^2-1}{c^2+1} \text{ and } b = \frac{2c}{c^2+1}$$

$$\Rightarrow \quad a^2+b^2=\left(\frac{c^2-1}{c^2+1}\right)^2+\frac{4c^2}{(c^2+1)^2} \text{ and } \frac{b}{a}=\left(\frac{2c}{c^2+1}\right)\Big/\left(\frac{c^2-1}{c^2+1}\right)$$

$$\Rightarrow \quad a^2+b^2=\frac{(c^2+1)^2}{(c^2+1)^2}=1 \text{ and } \frac{b}{a}=\frac{2c}{c^2-1}$$

EXAMPLE 12 *If $(x+iy)^{1/3}=a+ib$, $x, y, a, b \in R$. Show that $\frac{x}{a}+\frac{y}{b}=4(a^2-b^2)$.* [NCERT]

SOLUTION We have,

$$(x+iy)^{1/3}=a+ib$$

$$\Rightarrow \quad (x+iy)=(a+ib)^3 \qquad \text{[On cubing both sides]}$$

$$\Rightarrow \quad x+iy=a^3+3a^2ib+3a\,i^2b^2+i^3b^3$$

$$\Rightarrow \quad x+iy=(a^3-3ab^2)+i(3a^2b-b^3)$$

$$\Rightarrow \quad x=a^3-3ab^2 \text{ and } y=3a^2b-b^3$$

$$\Rightarrow \quad \frac{x}{a}=a^2-3b^2 \text{ and } \frac{y}{b}=3a^2-b^2$$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{b}=a^2-3b^2+3a^2-b^2=4(a^2-b^2)$$

Type III ON CONJUGATE OF A COMPLEX NUMBER

EXAMPLE 13 *Multiply $3-2i$ by its conjugate.*

SOLUTION The conjugate of $3-2i$ is $3+2i$.

Hence, required product $=(3-2i)(3+2i)=9-4i^2=9+4=13$

ALITER Let $z=3-2i$. Then, $\bar{z}=3+2i$

$$\therefore \quad z\bar{z}=|z|^2$$

$$\Rightarrow \quad z\bar{z}=3^2+(-2)^2=13$$

EXAMPLE 14 *Find the conjugate of $\frac{1}{3+4i}$*

SOLUTION Let $z=\frac{1}{3+4i}$. Then,

$$z=\frac{1}{3+4i}\times\frac{3-4i}{3-4i}=\frac{3-4i}{9+16}=\frac{3}{25}-\frac{4}{25}i$$

$$\therefore \quad \bar{z}=\frac{3}{25}+\frac{4}{25}i$$

EXAMPLE 15 *Express the following complex numbers in the standard form. Also, find their conjugate:*

(i) $\frac{1-i}{1+i}$ (ii) $\frac{(1+i)^2}{3-i}$ (iii) $\frac{(2+3i)^2}{2-i}$

SOLUTION We have,

(i)
$$z=\frac{1-i}{1+i}=\frac{1-i}{1+i}\times\frac{1-i}{1-i}=\frac{(1-i)^2}{1^2-i^2}=\frac{1-2i+i^2}{1-i^2}=\frac{1-2i-1}{1+1}=0-i$$

$$\therefore \quad \bar{z}=0+i$$

(ii)
$$z=\frac{(1+i)^2}{3-i}=\frac{1+2i+i^2}{3-i}\times\frac{3+i}{3+i}=\frac{2i}{3-i}\times\frac{3+i}{3+i}=\frac{6i+2i^2}{9-i^2}$$

$$\Rightarrow \quad z=\frac{6i-2}{10}=-\frac{1}{5}+\frac{3}{5}i$$

$$\therefore \quad \bar{z}=-\frac{1}{5}-\frac{3}{5}i$$

(iii)
$$z=\frac{(2+3i)^2}{2-i}=\frac{4+12i+9i^2}{2-i}=\frac{4+12i-9}{2-i}\times\frac{2+i}{2+i}=\frac{-5+12i}{2-i}\times\frac{2+i}{2+i}$$

$$\Rightarrow \quad z=\frac{-22+19i}{4-i^2}=-\frac{22}{5}+\frac{19}{5}i$$

$$\therefore \quad \bar{z}=-\frac{22}{5}-\frac{19}{5}i$$

EXAMPLE 16 *Find real values of x and y for which the complex numbers $-3+ix^2y$ and x^2+y+4i are conjugate of each other.*

SOLUTION Since $-3+ix^2y$ and x^2+y+4i are complex conjugates. Therefore,

$$-3+ix^2y=\overline{x^2+y+4i}$$

$$\Rightarrow \quad -3+ix^2y=x^2+y-4i$$

$$\Rightarrow \quad -3=x^2+y \qquad \text{...(i)}$$

$$\text{and,} \quad x^2y=-4 \qquad \text{...(ii)}$$

$$\Rightarrow \quad -3=x^2-\frac{4}{x^2} \qquad \text{[Putting } y=-4/x^2 \text{ from (ii) in (i)]}$$

$$\Rightarrow \quad x^4+3x^2-4=0$$

$$\Rightarrow \quad (x^2+4)(x^2-1)=0$$

$$\Rightarrow \quad x^2-1=0 \qquad [\because x^2+4\neq 0 \text{ for any real } x]$$

$$\Rightarrow \quad x=\pm 1$$

From (ii), $y=-4$, when $x=\pm 1$.

Hence, $x=1, y=-4$ or, $x=-1, y=-4$

EXAMPLE 17 *Find the real numbers x and y if $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.* [NCERT]

SOLUTION We have,

$$(x-iy)(3+5i)=(3x+5y)+i(5x-3y)$$

It is given that $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.

$$\therefore \quad (x-iy)(3+5i)=\overline{-6-24i}$$

$$\Rightarrow \quad (3x+5y)+i(5x-3y)=-6+24i$$

$$\Rightarrow \quad 3x+5y=-6 \text{ and } 5x-3y=24 \qquad \text{[On equating real and imaginary parts]}$$

Solving these equations, we get $x=3, y=-3$.

EXAMPLE 18 *The sum and product of two complex numbers are real if and only if they are conjugate of each other.*

SOLUTION First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1=a+ib$ and $z_2=a-ib$. Then,

$$z_1+z_2=(a+ib)+(a-ib)=2a, \text{ which is real.}$$

$$\text{And,} \quad z_1z_2=(a+ib)(a-ib)=a^2-i^2b^2=a^2+b^2, \text{ which is real.}$$

$\Rightarrow \quad (2n)! = n!\,(n+1)(n+2)\dots(2n-1)(2n)$

Now,
$$\left.\begin{array}{l} n+1>n \\ n+2>n \\ n+3>n \\ \vdots \\ n+(n-1)>n \\ n+n>n \end{array}\right\} \Rightarrow (n+1)(n+2)(n+3)\dots(2n-1)(2n)>n^n$$

$\Rightarrow \quad n!\,(n+1)(n+2)\dots(2n-1)(2n) > n!\,n^n$

$\Rightarrow \quad (2n)! > n!\,n^n \Rightarrow n!\,n^n < (2n)!$

From (i) and (ii), we get

$$(n!)^2 \le n^n\,(n!) < (2n)!$$

EXAMPLE 10 *Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?*

SOLUTION Let $E_2(n)$ denote the index of 2 in n. Then,

$E_2(33!) = E_2(1.2.3.4.5.6\dots.32.33)$

$\Rightarrow \quad E_2(33!) = E_2(2.4.6.8\dots.30.32)$

$\Rightarrow \quad E_2(33!) = 16 + E_2(1.2.3\dots.15.16)$

$\Rightarrow \quad E_2(33!) = 16 + E_2(2.4.6\dots14.16)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + E_2(1.2.3\dots8)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + E_2(2.4.6.8)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + 4 + E_2(1.2.3.4)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + 4 + E_2(2.4) = 16 + 8 + 4 + 3 = 31.$

Thus, exponent of 2 in 33! is 31 i.e.

$$33! = 2^{31} \times \text{an integer}$$

This shows that 33! is divisible by 2^{15} and the largest integer n such that 33! is divisible by 2^n is 31.

EXERCISE 16.1

1. Compute:
   (i) $\dfrac{30!}{28!}$ (ii) $\dfrac{11! - 10!}{9!}$ (iii) L.C.M. (6!, 7!, 8!)
2. Prove that $\dfrac{1}{9!} + \dfrac{1}{10!} + \dfrac{1}{11!} = \dfrac{122}{11!}$
3. Find x in each of the following:
   (i) $\dfrac{1}{4!} + \dfrac{1}{5!} = \dfrac{x}{6!}$
4. Convert the following products into factorials:
   (i) $5\cdot6\cdot7\cdot8\cdot9\cdot10$ (ii) $3\cdot6\cdot9\cdot12\cdot15\cdot18$
   (iii) $(n+1)(n+2)(n+3)\dots(2n)$ (iv) $1\cdot3\cdot5\cdot7\cdot9\dots(2n-1)$
5. Which of the following are true:
   (i) $(2+3)! = 2! + 3!$ (ii) $(2\times3)! = 2!\times3!$
6. Prove that: $n!\,(n+2) = n! + (n+1)!$
7. If $(n+2)! = 60\,[(n-1)!]$, find n.
8. If $(n+1)! = 90\,[(n-1)!]$, find n.

9. If $(n+3)! = 56\,[(n+1)!]$, find n.
10. If $\dfrac{(2n)!}{3!\,(2n-3)!}$ and $\dfrac{n!}{2!\,(n-2)!}$ are in the ratio 44 : 3, find n.
11. Prove that:
    (i) $\dfrac{n!}{(n-r)!} = n(n-1)(n-2)\dots(n-(r-1))$
    (ii) $\dfrac{n!}{(n-r)!\,r!} + \dfrac{n!}{(n-r+1)!\,(r-1)!} = \dfrac{(n+1)!}{r!\,(n-r+1)!}$
12. Prove that: $\dfrac{(2n+1)!}{n!} = 2^n\,[1\cdot3\cdot5\dots(2n-1)(2n+1)]$

ANSWERS

1. (i) 870 (ii) 100 (iii) 8! 3. (i) 36 (ii) 100 (iii) 64
4. (i) $\dfrac{10!}{4!}$ (ii) $3^6\,(6!)$ (iii) $\dfrac{(2n)!}{n!}$ (iv) $\dfrac{(2n)!}{2^n\,n!}$
5. (i) False (ii) False 7. 3 8. 9 9. 5 10. 6

16.2 FUNDAMENTAL PRINCIPLES OF COUNTING

In this section, we shall discuss two fundamental principles viz. principle of addition and principle of multiplication. These two principles will enable us to understand permutations and combinations. In fact these two principles form the base of permutations and combinations.

FUNDAMENTAL PRINCIPLE OF MULTIPLICATION *If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in m × n ways.*

EXPLANATION If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

ILLUSTRATION 1 *In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection?*

SOLUTION Here the teacher is to perform two jobs:

(i) selecting a boy among 10 boys, and
(ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of multiplication, the required number of ways is $10 \times 8 = 80$.

REMARK *The above principle can be extended for any finite number of jobs as stated below: If there are n jobs $J_1, J_2, \dots J_n$ such that job J_i can be performed independently in m_i ways; $i = 1, 2, \dots, n$. Then the total number of ways in which all the jobs can be performed is $m_1 \times m_2 \times m_3 \times \dots \times m_n$.*

FUNDAMENTAL PRINCIPLE OF ADDITION *If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in (m + n) ways.*

ILLUSTRATION 2 *In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make this selection?*

SOLUTION Here the teacher is to perform either of the following two jobs :

(i) selecting a boy among 10 boys.

or, (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore, by fundamental principle of addition either of the two jobs can be performed in $(10 + 8) = 18$ ways. Hence, the teacher can make the selection of either a boy or a girl in 18 ways.

DIFFERENCE BETWEEN THE TWO PRINCIPLES *As we have discussed in the principle of multiplication a job is divided or decomposed into a number of sub-jobs which are unconnected to each other and the job is said to be performed if each sub-job is performed. While in the principle of addition there are a number of independent jobs and we have to perform one of them. So, the total number of ways of completing any one of the sub-jobs is the sum of the number of ways of completing each sub-jobs.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *There are 3 candidates for a Classical, 5 for a Mathematical, and 4 for a Natural science scholarship.*

(i) In how many ways can these scholarships be awarded?

(ii) In how many ways one of these scholarships be awarded?

SOLUTION Clearly, Classical scholarship can be awarded to any one of the three candidates. So, there are 3 ways of awarding the Classical scholarship.

Similarly, Mathematical and Natural science scholarships can be awarded in 5 and 4 ways respectively. So,

Number of ways of awarding three scholarships

$= 3 \times 5 \times 4 = 60$ [By Fundamental Principle of multiplication]

And, number of way of awarding one of the three scholarships

$= 3 + 5 + 4 = 12$

EXAMPLE 2 *A room has 6 doors. In how many ways can a man enter the room through one door and come out through a different door ?*

SOLUTION Clearly, a person can enter the room through any one of the six doors. So there are six ways of entering into the room. After entering into the room, the man can come out through any one of the remaining five doors. So, he can come out through a different door in 5 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door $= 6 \times 5 = 30$.

EXAMPLE 3 *The flag of a newly formed forum is in the form □□□ of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible ?*

SOLUTION Since there are six colours to choose from, therefore, first block can be coloured in 6 ways.

Now, the second block can be coloured by any one of the remaining colours in five ways. So, there are five ways to colour the second block.

After colouring first two blocks only four colours are left. The third block can now be coloured by any one of the remaining four colours. So, there are four ways to colour the third block.

Hence, by the fundamental principle of multiplication, the number of flag-designs is $6 \times 5 \times 4 = 120$.

EXAMPLE 4 *Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, when* **[NCERT]**

(i) the repetition of the letters is not allowed. (ii) the repetition of the letters is allowed.

SOLUTION (i) The total number of words is same as the number of ways of filling in 4 vacant places □□□□ by the 4 letters. The first place can be filled in 4 different ways by any one of the 4 letters R, O, S, E. Since the repetition of letters is not allowed. Therefore, the second place can be filled in by any one of the remaining 3 letters in 3 different ways, following which the third place can be filled in by the remaining 2 letters in 2 different ways; following which the fourth place can be filled in by the remaining one letter in one way. Thus, by the fundamental principle of counting the required number of ways is $4 \times 3 \times 2 \times 1 = 24$.

Hence, required number of words = 24.

(ii) If the repetition of the letters is allowed, then each of the 4 vacant places can be filled in succession in 4 different ways.

Hence, required number of words $= 4 \times 4 \times 4 \times 4 = 256$.

EXAMPLE 5 *Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?* **[NCERT]**

SOLUTION The total number of signals is equal to the number of ways of filling in 2 vacant places in succession by four flags of different colours. The upper cacant place can be filled in 4 different ways by any one of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by any one of the remaining the different flags.

Hence, by the fundamental principle of multiplication, the required number of signals is $4 \times 3 = 12$.

EXAMPLE 6 *Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.* **[NCERT]**

SOLUTION Since a signal may consist of either 2 flags, 3 flags, 4 flags or 5 flags. Therefore,

Total number of signals = Number of 2 flags signals

+ Nnumber of 3 flags signals

+ Number of 4 flags signals

+ Number of 5 flags signals

$= 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1$

$= 20 + 60 + 120 + 120 = 320$

EXAMPLE 7 *Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin (i) at any one of the 7 floors (ii) at different floors.*

SOLUTION Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$.

(iii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4 and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$.

EXAMPLE 8 *In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the text book. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can three questions be selected?*

SOLUTION There are 12 questions in exercise 7. So, one question from exercise 7 can be selected in 12 ways. Exercise 8 contains 18 questions. So, second question can be selected in 18 ways. There are 9 questions in exercise 9. So, third question can be selected in 9 ways. Hence, three questions can be selected in $12 \times 18 \times 9 = 1944$ ways.

EXAMPLE 9 *A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years?*

SOLUTION The mint has to perform two jobs, viz.

(i) selecting the number of days in the February month (there can be 28 days or 29 days), and

(ii) selecting the first day of the February month.

The first job can be completed in 2 ways while the second can be performed in 7 ways by selecting any one of the seven days of a week.

Thus, the required number of plates $= 2 \times 7 = 14$.

EXAMPLE 10 *How many words (with or without meaning) of three distinct letters of the English alphabets are there?*

SOLUTION Here we have to fill up three places by distinct letters of the English alphabets. Since there are 26 letters of the English alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place. Now, the second place can be filled up by any of the remaining 25 letters. So, there are 25 ways of filling up the second place. After filling up the first two places only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways.

Hence, the required number of words $= 26 \times 25 \times 24 = 15600$

EXAMPLE 11 *There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 5 each?*

SOLUTION Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$

EXAMPLE 12 *Find the total number of ways of answering 5 objective type questions, each question having 4 choices.*

SOLUTION Since each question can be answered in 4 ways. So, the total number of ways of answering 5 questions is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 13 *For a set of five true/false questions, no student has written all correct answers and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible?*

SOLUTION Since a true/false type question can be answered in 2 ways, either by marking it true or false. So, there are 2 ways of answering each of the 5 questions.

Total number of different sequences of answers $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$.

Out of these 32 sequences of answers there is only one sequence of answering all the five questions correctly. But no student has written all the correct answers and different students have given different sequences of answers.

Maximum number of students in the class

= Number of sequences except one sequence in which all answers are correct

$= 32 - 1 = 31$

EXAMPLE 14 *How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6?*

SOLUTION We have to determine the total number of three digit numbers formed by using the digits 1, 7, 8, 9. Clearly, the repetition of digits is allowed.

A three digit number has three places viz. units's, ten's and hundred's. Unit's place can be filled by any of the digits 1, 7, 8, 9. So, unit's place can be filled in 4 ways. Similarly, each one of the ten's and hundred's place can be filled in 4 ways.

Total number of required numbers $= 4 \times 4 \times 4 = 64$.

EXAMPLE 15 *How many numbers are there between 100 and 1000 in which all the digits are distinct?*

SOLUTION A number between 100 and 1000 has three digits. So, we have to form all possible 3-digit numbers with distinct digits.

We cannot have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits 1, 2, 3, ..., 9. So, there are 9 ways of filling the hundred's place.

Now, 9 digits are left including 0. So, ten's place can be filled with any of the remaining 9 digits in 9 ways. Now, the unit's place can be filled with in any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

Hence, the total number of required numbers $= 9 \times 9 \times 8 = 648$.

EXAMPLE 16 *How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?*

SOLUTION Every number between 100 and 1000 consists of three digits. So, we have to determine the total number of three digit numbers such that every digit is either 2 or 9. Clearly, each one of the unit's, ten's and hundred's place can be filled in 2 ways.

So, the total number of required numbers $= 2 \times 2 \times 2 = 8$.

EXAMPLE 17 *How many numbers are there between 100 and 1000 such that 7 is in the unit's place?*

SOLUTION Every number between 100 and 1000 is a three digit number. So, we have to form 3-digit numbers with 7 at the unit's place by using the digits 0, 1, 2, ..., 9. Clearly, repetition of digits is allowed. The hundred's place can be filled with any of the digits from 1 to 9 (zero cannot be there at hundred's place). So, hundred's place can be filled in 9 ways.

Now, the ten's place can be filled with any of the digits from 0 to 9. So, ten's place can be filled in 10 ways. Since all the numbers have digit 7 at the unit's place, so, unit's place can be filled in only one way.

Hence, by the fundamental principle of counting the total number of numbers between 100 and 1000 having 7 at the unit's place $= 9 \times 10 \times 1 = 90$.

EXAMPLE 18 *How many numbers are there between 100 and 1000 such that at least one of their digits is 7?*

SOLUTION Clearly, a number between 100 and 1000 has 3-digits

Total number of 3-digit numbers having at least one of their digits as 7

= (Total number of three-digit numbers) – (Total number of 3-digit numbers in which 7 does not appear at all)

Total number of three-digit numbers : We have to form three-digit numbers by using the digits 0, 1, 2, 3, ... 9.

Clearly, hundred's place can be filled in 9 ways and each of the ten's and one's place can be filled in 10 ways.

So, total number of 3-digit number $= 9 \times 10 \times 10 = 900$.

Total number of three-digit number in which 7 does not appear at all : Here we have to form three-digit numbers by using the digits 0 to 9, except 7.

So, hundred's place can be filled in 8 ways and each of the ten's and one's place can be filled in 9 ways. So, total number of three-digit numbers in which 7 does not appear at all is $8 \times 9 \times 9$.

Hence, total number of 3-digit numbers having at least one of their digits as 7 is $9 \times 10 \times 10 - 8 \times 9 \times 9 = 252$.

EXAMPLE 19 *How many numbers are there between 100 and 1000 which have exactly one of their digits as 7 ?*

SOLUTION A number between 100 and 1000 contains 3-digits. So, we have to form 3-digit numbers having exactly one of their digits as 7. Such type of numbers can be divided into three types:

(i) Those numbers that have 7 in the unit's place but not in any other place.

(ii) Those numbers that have 7 in the ten's place but not in any other place.

(iii) Those numbers that have 7 in the hundred's place but not in any other place.

Required number of numbers is the total number of these three types of numbers.

We shall now count these three types of numbers separately.

(i) *Those three-digit numbers that have 7 in the unit's place but not in any other place.*
The hundred's place can have any one of the digits from 0 to 9 except 0 and 7. So, hundred's place can be filled in 8 ways. The ten's place can have any one of the digits from 0 to 9 except 7. So, the number of ways the ten's place can be filled is 9. The unit's place has 7. So, it can be filled in only one way.
Thus, there are $8 \times 9 \times 1 = 72$ numbers of the first kind.

(ii) *Those three-digit numbers that have 7 in the ten's place but not in any other place.*
The number of ways to fill the hundred's place $= 8$
(by any one of the digits from 1, 2, 3, 4, 5, 6, 8, 9)
The number of ways to fill the ten's place $= 1$ (by 7 only)
The number of ways to fill the one's place $= 9$
(by any one of the digits 0, 1, 2, 3, 4, 5, 6, 8, 9)
Thus, there are $8 \times 1 \times 9 = 72$ numbers of the second kind.

(iii) *Those three-digit numbers that have 7 in the hundred's place but not at any other place.*
In this case, the hundred's place can be filled only in one way and each of the ten's and one's place can be filled in 9 ways.
So, there are $1 \times 9 \times 9 = 81$ numbers of the third kind.
Hence, the total number of required type of numbers $= 72 + 72 + 81 = 225$.

EXAMPLE 20 *A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards ?*

SOLUTION Since a card can be sent by any one of the three servants, so the number of ways of sending the invitation card to the first friend $= 3$. Similarly, invitation cards can be sent to each of the six friends in 3 ways.

So, the required number of ways $= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729$.

EXAMPLE 21 *How many three-digit numbers more than 600 can be formed by using the digits 2, 3, 4, 6, 7.*

SOLUTION Clearly, repetition of digits is allowed. Since a three-digit number greater than 600 will have 6 or 7 at hundred's place. So, hundred's place can be filled in 2 ways. Each of the ten's and one's place can be filled in 5 ways.

Hence, total number of required numbers $= 2 \times 5 \times 5 = 50$.

EXAMPLE 22 *A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made ?*

SOLUTION Since each arm can be kept in 4 positions and a signal is possible when all the 5 arms are simultaneously placed in positions. So,

Total number of ways of placing the arms $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

But this includes one inadmissible case, when all the arms are in the position of rest and then no signal can be made.

Hence, required number of signals $= (4^5 - 1) = 1023$.

EXAMPLE 23 *How many numbers between 3000 and 4000 can be formed from the digits 3, 4, 5, 6, 7 and 8, no digit being repeated in any number ?*

SOLUTION Clearly, a number between 3000 and 4000 must have 3 at thousand's place. So, thousand's place can be filled in only one way. Now, hundred's place can be filled in 5 ways. Since repetition of digits is not allowed so ten's and one's places can be filled in 4 and 3 ways respectively.

So, total number of required numbers $= 1 \times 5 \times 4 \times 3 = 60$.

EXAMPLE 24 *How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8.*

SOLUTION Clearly, a number between 4000 and 5000 must have 4 at thousand's place. Since the number is divisible by 5 it must have 5 at unit's place. Now, each of the remaining places (viz. hundred's and ten's) can be filled in 5 ways.

Hence, total number of required numbers $= 1 \times 5 \times 5 \times 1 = 25$.

EXAMPLE 25 *How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits is not allowed (ii) repetition of digits is allowed ?*

SOLUTION (i) In a four-digit number 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. (viz. 1, 2, 3, 4, 5). Since repetition of digits is not allowed and 0 can be used at hundred's place, so hundred's place can be filled in 5 ways. Now, any one of the remaining four digits can be used to fill up ten's place. So, ten's place can be filled in 4 ways. One's place can be filled from the remaining three digits in 3 ways.

Hence, the required number of numbers $= 5 \times 5 \times 4 \times 3 = 300$.

(ii) For a four-digit number we have to fill up four places and 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. Since repetition of digits is allowed, so each of the remaining three places viz. hundred's, ten's and one's can be filled in 6 ways.

Hence, the required number of numbers $= 5 \times 6 \times 6 \times 6 = 1080$.

EXAMPLE 26 *How many three-letter words can be formed using a, b, c, d, e if : (i) repetition is not allowed (ii) repetition is allowed ?*

SOLUTION (i) Clearly, the total number of three-letter words is equal to the number of ways of filling three places. First place can be filled in 5 ways. Now, four letters are left. So, the second place can be filled in 4 ways. Since the repetition of letters is not allowed, so the third place can be filled from any one of the remaining 3 digits in 3 ways.

Hence, total number of words $= 5 \times 4 \times 3 = 60$.

(ii) In this case repetition of letters is allowed, so each of the three places can be filled in 5 ways.

Hence, total number of words $= 5 \times 5 \times 5 = 125$.

EXAMPLE 27 *In how many ways can the following prizes be given away to a class of 30 students, first and second in Mathematics, first and second in Physics, first in Chemistry and first in English ?*

SOLUTION Here we have to give prizes in four subjects and the process of distributing prizes can be completed by giving prizes in the four subjects.

First and second prizes can be given in Mathematics in (30×29) ways.

First and second prizes can be given in Physics in (30×29) ways.

First prize can be given in Chemistry in 30 ways.

First prize can be given in English in 30 ways.

Hence, the number of ways to give prizes in all the four subjects

$$= (30 \times 29) \times (30 \times 29) \times 30 \times 30 = 6.8121 \times 10^8.$$

EXAMPLE 28 *How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4 if : (i) repetition of digits is allowed ? (ii) repetition of digits is not allowed ?*

SOLUTION (i) Every number between 1000 and 4000 is a four digit number. In thousand's place we can put either 1 or 2 or 3 but not 4. So, thousand's place can be filled in 3 ways. Since repetition of digits is allowed, so each of the hundred's, ten's and unit's place can be filled in 5 ways. So, total number of numbers between 1000 and 4000 including 1000 and excluding 4000 is $3 \times 5 \times 5 \times 5 = 375$. But we have to find the total number of numbers greater than 1000 but not greater than 4000.

Hence, required number of numbers $= 375 + 1$ (for 4000) $- 1$ (for 1000) $= 375$.

(ii) As discussed above thousand's place can be filled in 3 ways. Since repetition of digits is not allowed, so, hundred's place can be filled from the remaining digits in 4 ways. Now, three digits are left, so ten's place can be filled in 3 ways. One's place can be filled in 2 ways.

Hence, required number of numbers $= 3 \times 4 \times 3 \times 2 = 72$.

EXAMPLE 29 *How many three digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if :* [NCERT]

(i) *the repetition of digits is not allowed ?*

(ii) *the repetition of digits is allowed ?*

SOLUTION For a number to be odd, we must have 1, 3 or 5 at the unit's place. So, there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways. Now, four digits are left. So, hundred's place can be filled in 4 ways.

So, required number of numbers $= 3 \times 5 \times 4 = 60$

(ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence, required number of numbers $= 3 \times 6 \times 6 = 108$.

EXAMPLE 30 *How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?* [NCERT]

SOLUTION For a number to be even, we must have 2, 4 or 6 at the unit's place. So, there are 3 ways to fill in the unit's place. Since digits can be repeated, so each of the ten's and hundred's place can be filled in 6 ways.

EXAMPLE 31 *How many numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated ?*

SOLUTION The unit's place can be filled in 5 ways. Since, the repetition of digits is allowed, therefore ten's place can be filled in 5 ways and hundred's place can also be filled in 5 ways. Therefore, by the fundamental principle of counting, the required number of three digit numbers $= 5 \times 5 \times 5 = 125$.

EXAMPLE 32 *In how many ways 5 rings of different types can be worn in 4 fingers ?*

SOLUTION The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each one of the other rings can be worn in 4 ways.

Hence, the requisite number of ways $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 33 *Find the number of numbers of 5 digits that can be formed with the digits 0, 1, 2, 3, 4 if the digits can be repeated in the same number.*

SOLUTION In a five digit number 0 cannot be put in ten thousand's place. So, the number of ways of filling up the ten thousand's place $= 4$.

Since the repetition of digits is allowed, therefore each of the other places can be filled in 5 ways.

So, the required number of numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$.

EXAMPLE 34 *In how many ways can 3 prizes be distributed among 4 boys, when*

(i) *no boy gets more than one prize ?*

(ii) *a boy may get any number of prizes ?*

(iii) *no boy gets all the prizes ?*

SOLUTION (i) The first prize can be given away in 4 ways as it may be given to any one of the 4 boys. The second prize can be given away in 3 ways, because the boy who got the first prize cannot receive the second prize. The third prize can be given away to anyone of the remaining 2 boys in 2 ways.

So, the number of ways in which all the prizes can be given away $= 4 \times 3 \times 2 = 24$.

ALITER The total number of ways is the number of arrangements of 4 taken 3 at a time.

So, the requisite number of ways $= {}^4P_3 = 4! = 24$.

(ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys. The second prize can also be given away in 4 ways, since it may be obtained by the boy who has already received a prize. Similarly, third prize can be given away in 4 ways.

Hence, the number of ways in which all the prizes can be given away

$$= 4 \times 4 \times 4 = 4^3 = 64.$$

(iii) Since any one of the 4 boys may get all the prizes. So, the number of ways in which a boy gets all the 3 prizes is 4.

So, the number of ways in which a boy does not get all the prizes $= 64 - 4 = 60$.

EXAMPLE 35 *How many 4-digit numbers are there, when a digit may be repeated any number of times ?*

SOLUTION In a four digit number 0 cannot be placed at thousand's place. So, thousand's place can be filled with any digit from 1 to 9. Thus, thousand's place can be filled in 9 ways.

Since repetition of digits is allowed, therefore each of the remaining 3 places can be filled in 10 ways by using the digits from 0 to 9.

Hence, the required number of numbers $= 9 \times 10 \times 10 \times 10 = 9000$.

EXAMPLE 36 *In how many ways can 5 letters be posted in 4 letter boxes ?*

SOLUTION Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways.

So, total number of ways in which all the five letters can be posted is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 37 *Find the total number of ways in which n distinct objects can be put into two different boxes.*

SOLUTION Let the two boxes be B_1 and B_2. We observe that there are two choices for each of the n objects. Therefore, by fundamental priniciple of counting

$$\text{Total number of ways} = \underbrace{2 \times 2 \times \ldots \times 2}_{n-\text{times}} = 2^n$$

EXAMPLE 38 *Find the total number of ways in which n-distinct objects can be put into two different boxes so that no box remains empty.*

SOLUTION Each object can be put either in box B_1 (say) or in box B_2 (say). So, there are two choices for each of the n objects. Therefore, the number of choices for n distinct objects is

$$\underbrace{2 \times 2 \times \ldots \times 2}_{n-\text{times}} = 2^n$$

Two of these choices correspond to either the first or the second box being empty. Thus, there are $2^n - 2$ ways in which neither box is empty.

EXAMPLE 39 *By using the digits 0, 1, 2, 3, 4 and 5 (repetitions not allowed) numbers are formed by using any number of digits. Find the total number of non-zero numbers that can be formed.*

SOLUTION We have,

Required number of numbers

= Number of 1 digit number + No. of 2 digit numbers + ...+ Number of 6 digit numbers

$= 5 + 5 \times 5 + 5 \times 5 \times 4 + 5 \times 5 \times 4 \times 3 + 5 \times 5 \times 4 \times 3 \times 2 + 5 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5 + 25 + 100 + 300 + 600 + 600 = 1630.$

EXERCISE 16.2

1. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection ?
2. A person wants to buy one fountain pen, one ball pen and one pencil from a stationery shop. If there are 10 fountain pen varieties, 12 ball pen varieties and 5 pencil varieties, in how many ways can he select these articles ?
3. From Goa to Bombay there are two roots; air, and sea. From Bombay to Delhi there are three routes; air, rail and road. From Goa to Delhi via Bombay, how many kinds of routes are there ?
4. A mint prepares metallic calenders specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of calendars should it prepare to serve for all the possibilities in future years ?
5. There are four parcels and five post-offices. In how many different ways can the parcels be sent by registered post ?
6. A coin is tossed five times and outcomes are recorded. How many possible outcomes are there ?
7. In how many ways can an examinee answer a set of ten true/false type questions?
8. A letter lock consists of three rings each marked with 10 different letters. In how many ways it is possible to make an unsuccessful attempt to open the lock ?
9. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 2 each?

10. There are 5 books on Mathematics and 6 books on Physics in a book shop. In how many ways can a student buy : (i) a Mathematics book and a Physics book (ii) either a Mathematics book or a Physics book ?
11. Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other ? [NCERT]
12. A team consists of 6 boys and 4 girls and other has 5 boys and 3 girls. How many single matches can be arranged between the two teams when a boy plays against a boy and a girl plays against a girl ?
13. Twelve students compete in a race. In how many ways first three prizes be given ?
14. How many A.P.'s with 10 terms are there whose first term is in the set {1, 2, 3} and whose common difference is in the set {1, 2, 3, 4, 5} ?
15. From among the 36 teachers in a college, one principal, one vice-principal and the teacher-incharge are to be appointed. In how many ways can this be done ?
16. How many three-digit numbers are there with no digit repeated ?
17. How many three-digit numbers are there ?
18. How many three-digit odd numbers are there ?
19. How many different five-digit number licence plates can be made if
    (i) first digit cannot be zero and the repetition of digits is not allowed,
    (ii) the first-digit cannot be zero, but the repetition of digits is allowed ?
20. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 7000, if repetition of digits is not allowed ?
21. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 8000, if repetition of digits is not allowed ?
22. In how many ways can six persons be seated in a row ?
23. How many 9-digit numbers of different digits can be formed ?
24. How many odd numbers less than 1000 can be formed by using the digits 0, 3, 5, 7 when repetition of digits is not allowed ?
25. How many 3-digit numbers are there, with distinct digits, with each digit odd ?
26. How many different numbers of six digits each can be formed from the digits 4, 5, 6, 7, 8, 9 when repetition of digits is not allowed ?
27. How many different numbers of six digits can be formed from the digits 3, 1, 7, 0, 9, 5 when repetition of digits is not allowed ?
28. How many four digit different numbers, greater than 5000 can be formed with the digits 1, 2, 5, 9, 0 when repetition of digits is not allowed ?
29. Serial numbers for an item produced in a factory are to be made using two letters followed by four digits (0 to 9). If the letters are to be taken from six letters of English alphabet without repetition and the digits are also not repeated in a serial number, how many serial numbers are possible ?
30. A number lock on a suitcase has 3 wheels each labelled with ten digits 0 to 9. If opening of the lock is a particular sequence of three digits with no repeats, how many such sequences will be possible ? Also, find the number of unsuccessful attempts to open the lock.
31. A customer forgets a four-digit code for an Automatic Teller Machine (ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6 and 9. Find the largest possible number of trials necessary to obtain the correct code.

32. In how many ways can three jobs I, II and III be assigned to three persons *A*, *B* and *C* if one person is assigned only one job and all are capable of doing each job ?
33. How many four digit natural numbers not exceeding 4321 can be formed with the digits 1, 2, 3 and 4, if the digits can repeat ?
34. How many numbers of six digits can be formed from the digits 0, 1, 3, 5, 7 and 9 when no digit is repeated ? How many of them are divisible by 10 ?
35. If three six faced die each marked with numbers 1 to 6 on six faces, are thrown find the total number of possible outcomes.
36. A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there ? How many possible outcomes if the coin is tossed four times? Five times? *n* times?
37. Find the number of ways in which 8 distinct toys can be distributed among 5 children.
38. Find the number of ways in which one can post 5 letters in 7 letter boxes.
39. Three dice are rolled. Find the number of possible outcomes in which at least one die shows 5.
40. Find the total number of ways in which 20 balls can be put into 5 boxes so that first box contains just one ball.
41. In how many ways can 5 different balls be distributed among three boxes ?
42. In how many ways can 7 letters be posted in 4 letter boxes ?
43. In how many ways can 4 prizes be distributed among 5 students, when
    (i) no student gets more than one prize ?
    (ii) a student may get any number of prizes ?
    (iii) no student gets all the prizes ?
44. How many numbers of four digits can be formed with the digits 1, 2, 3, 4, 5 if the digits can be repeated in the same number ?
45. How many three digit numbers can be formed by using the digits 0, 1, 3, 5, 7 while each digit may be repeated any number of times ?
46. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times ?
47. A coin is tossed 3 times and the outcomes are recorded. How many possible outcomes are there? [NCERT]
48. How many five digit telephone numbers can be constructed using the digits 0 to 9. If each number starts with 67 and no digit appears more than once? [NCERT]

ANSWERS

1. 378	2. 600	3. 6	4. 14	5. 625	6. 32
7. 1024	8. 999	9. 512	10. (i) 30	(ii) 11	11. 42
12. 42	13. 1320	14. 15	15. 42840	16. 648	17. 900
18. 450	19. (i) 27216 (ii) 90000	20. 72	21. 48		22. 720
23. 9(9!)	24. 21	25. 60	26. 720	27. 600	28. 48
29. 151200	30. 720, 719	31. 24	32. 6	33. 229	34. 600, 120
35. 216	36. 8, 16, 2^n	37. 5^8	38. 7^5	39. 91	40. 20×4^{19}

41. 243	42. 4^7	43. (i) 5!	(ii) 625	(iii) 620	44. 625
45. 100	46. 215	47. 8	48. 336		

HINTS TO SELECTED PROBLEMS

1. No. of ways $= 27 \times 14$.
2. Required number of ways $= 10 \times 12 \times 5 = 600$.
3. No of routes $= 2 \times 3 = 6$.
4. Total number of calendars $= 7 \times 2 = 14$.
5. Since a parcel can be sent to any one of the five post offices. So, required number of ways $= 5 \times 5 \times 5 \times 5 = 5^4$.
6. Since toss of each coin can result in 2 ways. So, required no. of ways $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5$.
8. Required no. of ways $= 10 \times 10 \times 10 - 1$.
9. Each one of the first three questions can be answered in 4 ways and each one of the next three questions can be answered in 2 ways. So, total no. of sequences of answers $= 4 \times 4 \times 4 \times 2 \times 2 \times 2$.
11. Required no. of signals $= 7 \times 6$.
12. A boy can be selected from the first team in 6 ways, and from the second in 5 ways. So, no. of single matches between the boys of two teams $= 6 \times 5 = 30$. Similarly, the no. of single matches between the girls of two teams $= 4 \times 3 = 12$. So, total number of matches $= 30 + 12 = 42$.
13. Required no. of ways $= 12 \times 11 \times 10$.
14. There are 3 ways to choose the first term and corresponding to each such way there are 5 ways of selecting the common difference. So, required no. of A.P.'s $= 3 \times 5$.
15. Required no. of ways $= 36 \times 35 \times 34$.
16. The total no. of required numbers $= 9 \times 9 \times 8$.
17. The total no. of required numbers $= 9 \times 10 \times 10$.
18. The total no. of required number $= 9 \times 10 \times 5$.
19. (i) Required no. of licence plates $= 9 \times 9 \times 8 \times 7 \times 6$
    (ii) Required no. of licence plates $= 9 \times 10 \times 10 \times 10 \times 10$.
20. Required no. of numbers $= 3 \times 4 \times 3 \times 2$.
21. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.
22. Required no. of ways $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.
23. Required no. of numbers $= 9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$.
24. An odd number less than 1000 may be a one-digit number, two-digit number or a three-digit number. So, required no. of numbers is
    3 (one-digit nos.) $+ 2 \times 3$ (two-digit nos.) $+ 2 \times 2 \times 3$ (3-digit nos.)
25. Required no. of numbers $= 5 \times 4 \times 3$.
26. Required no. of numbers $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.
27. Required no. of numbers $= 5 \times 5 \times 4 \times 3 \times 2 \times 1$.
28. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.
29. Here we have to perform 6 jobs. So, required number of serial numbers is $6 \times 5 \times 10 \times 9 \times 8 \times 7$

30. Required number of sequences $= 10 \times 9 \times 8$.
Also, total number of unsuccessful attempts $= 10 \times 9 \times 8 - 1$
31. Number of trials $= 4 \times 3 \times 2 \times 1$
32. Required number of ways $= 3 \times 2 \times 1$
36. Since a toss of a coin can result in a head or a tail. Therefore, if a coin is tossed n-times, then the total number of outcomes is

$$\underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n\text{-times}} = 2^n$$

37. Each toy can be distributed in 5 ways. So, total number of ways

$$= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^8$$

38. Each letter can be posted in any one of the 7 letter boxes. So, required number of ways $= 7 \times 7 \times 7 \times 7 \times 7 = 7^5$
39. Required number of possible outcomes
= Total number of possible outcomes – Number of possible out- comes in which 5 does not appear on any dice.

$$= 6^3 - 5^3 = 216 - 125 = 91.$$

40. One ball can be put in first box in 20 ways because we can put any one of the twenty balls in the first box. Now, remaining 19 balls are to be put into remaining 4 boxes. This can be done in 4^{19} ways, because there are 4 choices for each ball. Hence, the required number of ways $= 20 \times 4^{19}$.
47. Required number of outcomes $= 2 \times 2 \times 2$
48. Required number of telephone numbers $= 8 \times 7 \times 6$

16.3 PERMUTATIONS

Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.

For example, if there are three objects, then the permutations of these objects, taking two at a time, are

$$ab, \quad ba, \quad bc, \quad cb, \quad ac, \quad ca$$

So, the number of permutations of three different things taken two at a time is 6.

NOTE *It should be noted that in permutations the order of arrangement is taken into account, when the order is changed, a different permutation is obtained.*

ILLUSTRATION 1 *Write down all the permutations of the set of three letters A, B, C.*

SOLUTION The permutations of three letters *A, B, C* taking all at a time are :

ABC, ACB, BCA, BAC, CBA, CAB.

Clearly, there are 6 permutations.

ILLUSTRATION 2 *Write down all the permutations of the vowels A, E, I, O, U in English alphabets taking three at a time, and starting with A.*

SOLUTION The permutations of vowels *A, E, I, O, U* taking three at a time, and starting with A are

AEI, AIE, AEO, AOE, AEU, AUE, AIO, AOI, AIU, AUI, AOU, AUO

Clearly, there are 12 permutations.

ILLUSTRATION 3 *Write down all the permutations of letters A, B, C, D taking three at a time.*

SOLUTION The desired permutations are :

ABC	ABD	BCD	ACD
ACB	ADB	BDC	ADC
BCA	BDA	CBD	CAD
BAC	BAD	CDB	CDA
CAB	DAB	DCB	DAC
CBA	DBA	DBC	DCA

Clearly, there are 24 permutations. These permutations are obtained by first selecting three letters out of 4 and then arranging them in all possible ways.

A NOTATION *If n and r are positive integers such that $1 \le r \le n$, then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$.*

Thus,

${}^nP_r$ or, $P(n, r)$ = Total number of permutations of n distinct things, taken r at a time.

In illustration 3, we have seen that there are 24 permutations, on a set of 4 letters, taken 3 at a time. Therefore, as per our notation, we have

$${}^4P_3 = 24 \text{ or, } P(4, 3) = 24.$$

THEOREM 1 *Let r and n be positive integers such that $1 \le r \le n$. Then the number of all permutations of n distinct things taken r at a time is given by*

$$n(n-1)(n-2)(n-3)\ldots(n-(r-1)).$$

i.e. $\quad P(n, r) = {}^nP_r = n(n-1)(n-2)\ldots(n-(r-1)).$

PROOF The number of permutations of n distinct things, taken r at a time, is same as the number of ways in which we can fill up r-places when we have n different things at our disposal.

The first place can be filled in n ways, for any one of the n things can be used to fill up the first place. Having filled it, there are $(n-1)$ things left and any one of these $(n-1)$ things can be used to fill up the second place. So, the second place can be filled in $(n-1)$ ways. Hence, by the fundamental principle of counting, the first two places can be filled in $n(n-1)$ ways. When the first two places are filled, there are $(n-2)$ places left, so that the third place can be filled from the remaining $(n-2)$ things in $(n-2)$ ways. Therefore, the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing in this manner, we find that the first $(r-1)$ places can be filled in

$$n(n-1)(n-2)\ldots(n-(r-2)) \text{ ways.}$$

After filling up first $(r-1)$ places, exactly $n-(r-1) = n-r+1$ things are left. So, the rth place can be filled in $(n-(r-1))$ ways. Hence, the r places can be filled in $n(n-1)(n-2)\ldots(n-(r-1))$ ways.

Hence, the total number of permutations of n distinct things, taken r at a time is

$$n(n-1)(n-2)(n-3)\ldots(n-(r-1)).$$

Thus, $\quad P(n, r) = n(n-1)(n-2)(n-3)\ldots(n-(r-1)).$

THEOREM 2 *Prove that:* $P(n, r) = {}^nP_r = \dfrac{n!}{(n-r)!}$

PROOF We have,

$$P(n, r) = n(n-1)(n-2)(n-3)\ldots(n-(r-1))$$

$\Rightarrow \quad P(n, r) = \frac{n(n-1)(n-2)(n-3)\dots(n-(r-1))(n-r)(n-(r+1))\dots 3.2.1}{(n-r)(n-(r+1))\dots 3.2.1}$

$\Rightarrow \quad P(n, r) = \frac{n!}{(n-r)!}$

THEOREM 3 *The number of all permutations of n distinct things, taken all at a time is n!.*

PROOF The number of all permutations of n distinct things, taken all at a time is same as the number of ways of filling n places when we have n distinct things at our disposal.

Proceeding as in theorem 1, we have

$P(n, n) = n(n-1)(n-2)(n-3)\dots(n-(n-1))$

$\Rightarrow \quad P(n, n) = n(n-1)(n-2)(n-3)\dots 3.2.1 = n!$

THEOREM 4 *Prove that $0! = 1$.*

PROOF We have,

$P(n, r) = \frac{n!}{(n-r)!}$

$\Rightarrow \quad P(n, n) = \frac{n!}{0!}$ [Putting $r = n$]

$\Rightarrow \quad n! = \frac{n!}{0!}$ [$\because P(n, n) = n!$ (See Theorem 3)]

$\Rightarrow \quad 0! = \frac{n!}{n!} = 1.$

ILLUSTRATIVE EXAMPLES

Type I **PROBLEMS BASED UPON THE VALUE OF ^nP_r or, $P(n, r)$**

EXAMPLE 1 *Evaluate the following :*

(i) $^5P_3$ (ii) $P(15, 3)$ (iii) $P(5, 5)$

SOLUTION We have,

(i) $^5P_3 = \frac{5!}{(5-3)!}$ $\left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$

$\Rightarrow \quad {}^5P_3 = \frac{5!}{2!} = \frac{5\times4\times3\times2!}{2!} = 60$

(ii) $P(15, 3) = \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{15\times14\times13\times12!}{12!} = 2730$

(iii) $P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5! = 120$

Type II **ON FINDING THE VALUE OF REQUIRED UN-KNOWN WHEN A RELATION CONNECTING $P(n, r)$ IS GIVEN**

EXAMPLE 2 *If $2.P(5, 3) = P(n, 4)$, find n.*

SOLUTION We have,

$2.P(5, 3) = P(n, 4)$

$\Rightarrow \quad P(n, 4) = 2.P(5, 3)$

$\Rightarrow \quad \frac{n!}{(n-4)!} = 2\left(\frac{5!}{(5-3)!}\right)$

$\Rightarrow \quad \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = \frac{2(5!)}{2!}$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5!$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5\times4\times3\times2\times1$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5\cdot(5-1)\cdot(5-2)(5-3)$

$\Rightarrow \quad n = 5$ [By comparing two sides]

EXAMPLE 3 *If $P(n, 4) = 20\times P(n, 2)$, find n.*

SOLUTION We have,

$P(n, 4) = 20\times P(n, 2)$

$\Rightarrow \quad \frac{n!}{(n-4)!} = 20\times\frac{n!}{(n-2)!}$

$\Rightarrow \quad (n-2)! = 20\times(n-4)! \Rightarrow (n-2)(n-3)(n-4)! = 20\times(n-4)!$

$\Rightarrow \quad (n-2)(n-3) = 20 \quad \Rightarrow (n-2)(n-3) = 5\times4$ [By comparing two sides]

$\Rightarrow \quad n-3 = 4 \Rightarrow n = 7$

EXAMPLE 4 *If $P(5, r) = 2.P(6, r-1)$, find r.* [NCERT]

SOLUTION We have,

$P(5, r) = 2.P(6, r-1)$

$\Rightarrow \quad \frac{5!}{(5-r)!} = 2.\frac{6!}{(6-(r-1))!}$

$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{2\times6\times5!}{(7-r)!}$

$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{12\times5!}{(7-r)(6-r)(5-r)!}$

$\Rightarrow \quad 1 = \frac{12}{(7-r)(6-r)}$

$\Rightarrow \quad (7-r)(6-r) = 12$

$\Rightarrow \quad (7-2)(6-r) = 4\times3$

$\Rightarrow \quad 7-r = 4$ [By comparing]

$\Rightarrow \quad r = 3$

EXAMPLE 5 *If $^{10}P_r = 5040$, find the value of r.*

SOLUTION We have,

$^{10}P_r = 5040$

$\Rightarrow \quad \frac{10!}{(10-r)!} = 10\times504$

$\Rightarrow \quad \frac{10!}{(10-r)!} = 10\times9\times8\times7$

$\Rightarrow \quad \frac{10!}{(10-r)!} = \frac{10\times9\times8\times7\times6!}{6!}$

$\Rightarrow \quad \frac{10!}{(10-r)!} = \frac{10!}{6!}$

$\Rightarrow \quad (10-r)! = 6! \Rightarrow 10-r = 6 \Rightarrow r = 4$

EXAMPLE 6 *If $P(n-1, 3) : P(n, 4) = 1 : 9$, find n.*

SOLUTION We have,

$P(n-1, 3) : P(n, 4) = 1 : 9$

$\Rightarrow \quad \frac{P(n-1, 3)}{P(n, 4)} = \frac{1}{9}$

$$\Rightarrow \quad \frac{\frac{(n-1)!}{(n-1-3)!}}{\frac{n!}{(n-4)!}} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n!} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{(n-1)!}{n!} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{(n-1)!}{n\cdot(n-1)!} = \frac{1}{9} \Rightarrow n = 9$$

EXAMPLE 7 *If* ${}^9P_5 + 5\cdot {}^9P_4 = {}^{10}P_r$, *find r.*

SOLUTION We have,

$${}^9P_5 + 5\cdot{}^9P_4 = {}^{10}P_r$$

$$\Rightarrow \quad \frac{9!}{(9-5)!} + 5\cdot\frac{9!}{(9-4)!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + 5\cdot\frac{9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + \frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad 2\times\frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{5\times 2\times 9!}{5\times 4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10\times 9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad (10-r)! = 5! \Rightarrow 10-r = 5 \Rightarrow r = 5$$

EXAMPLE 8 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$, *find r.*

SOLUTION We have,

$${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$$

$$\Rightarrow \quad \frac{56!}{(56-r-6)!} : \frac{54!}{(54-r-3)!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} : \frac{54!}{(51-r)!} = 30800 : 1$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56\times 55\times 54!}{(50-r)!} \times \frac{(51-r)\times(50-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad 56\times 55\times(51-r) = 30800$$

$$\Rightarrow \quad (51-r) = 10 \Rightarrow r = 41.$$

EXAMPLE 9 *If* ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$, *find n.*

SOLUTION We have,

$${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$$

$$\frac{{}^{2n+1}P_{n-1}}{{}^{2n-1}P_n} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)(2n)(2n-1)!}{(n+2)(n+1)\,n\,(n-1)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{2(2n+1)}{(n+2)(n+1)} = \frac{3}{5}$$

$$\Rightarrow \quad 10(2n+1) = 3(n+2)(n+1)$$

$$\Rightarrow \quad 3n^2 + 9n + 6 = 20n + 10$$

$$\Rightarrow \quad 3n^2 - 11n - 4 = 0$$

$$\Rightarrow \quad (n-4)(3n+1) = 0 \Rightarrow n = 4 \qquad [\because n \neq -1/3]$$

EXAMPLE 10 *If* ${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, *find r.*

SOLUTION We have,

$${}^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} : \frac{20!}{(18-r)!} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22\times 21\times 20!}{(21-r)(20-r)(19-r)\cdot(18-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22\times 21}{(21-r)(20-r)(19-r)} = \frac{11}{52}$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2\times 21\times 52$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2\times 3\times 7\times 4\times 13$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 12\times 13\times 14$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = (21-7)(20-7)(19-7)$$

$$\Rightarrow \quad r = 7$$

Type III **ON PROVING RESULTS RELATED TO** $P(n, r)$ **or** ${}^nP_r$

EXAMPLE 11 *Prove the following:*

(i) $P(n, n) = 2\,P(n, n-2)$

(ii) $P(n, n) = P(n, n-1)$

(iii) $P(n, r) = P(n-1, r) + r\,.\,P(n-1, r-1)$

(iv) $P(n, r) = n\cdot P(n-1, r-1)$

SOLUTION We have,

(i) $$2P(n, n-2) = 2\,\frac{n!}{(n-(n-2))!} = 2\left(\frac{n!}{2!}\right) = n! = P(n, n)$$

(ii) $$P(n, n-1) = \frac{n!}{(n-(n-1))!} = \frac{n!}{1!} = n! = P(n, n)$$

(iii) $$P(n-1, r) + r\cdot P(n-1, r-1)$$

$$= \frac{(n-1)!}{(n-1-r)!} + r\cdot\frac{(n-1)!}{((n-1)-(r-1))!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!} = \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left\{1 + \frac{r}{n-r}\right\} = \frac{(n-1)!}{(n-r-1)!}\left(\frac{n-r+r}{n-r}\right)$$

$$= \frac{(n-1)!}{(n-r-1)!}\cdot\frac{n}{n-r} = \frac{n!}{(n-r)!} = P(n, r)$$

(iv) $$n \cdot P(n-1, r-1) = n\frac{(n-1)!}{((n-1)-(r-1))!} = \frac{n!}{(n-r)!} = P(n, r)$$

EXAMPLE 12 *Prove that if $r \le s \le n$, then $P(n, s)$ is divisible by $P(n, r)$.*

SOLUTION Let $s = r + k$ where $0 \le k \le s - r$. Then,

$$P(n, s) = \frac{n!}{(n-s)!} = n(n-1)(n-2)\dots(n-(s-1))$$

$\Rightarrow$ $P(n, s) = n(n-1)(n-2)\dots(n-(r+k-1))$

$\Rightarrow$ $P(n, s) = n(n-1)(n-2)\dots(n-(r-1))(n-r)(n-(r+1))\dots(n-(r+k-1))$

$\Rightarrow$ $P(n, s) = \{n(n-1)(n-2)\dots n-(r-1)\}\{(n-r)(n-(r+1))\dots(n-(r+k-1))\}$

$\Rightarrow$ $P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\dots(n-(r+k-1))\}$

$$\left[\because P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\dots(n-(r-1))\right]$$

$\Rightarrow$ $P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\dots(n-(r+k-1))\}$

$\Rightarrow$ $P(n, s)$ is divisible by $P(n, r)$.

EXAMPLE 13 *If P_m stands for ${}^mP_m$, then prove that:*

$$1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \dots + n\cdot P_n = (n+1)!$$

SOLUTION We have, $P_m = {}^mP_m = m!$.

So, $1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \dots + n\cdot P_n$

$$= 1 + 1 + 2\cdot 2! + 3\cdot 3! + 4\cdot 4! + \dots + n\cdot n!$$

$$= 1 + \sum_{r=1}^{n} r\cdot r!$$

$$= 1 + \sum_{r=1}^{n} \{(r+1) - 1\}\, r!$$

$$= 1 + \sum_{r=1}^{n} [(r+1)\, r! - r!]$$

$$= 1 + \sum_{r=1}^{n} [(r+1)! - r!]$$

$$= 1 + [(2! - 1!) + (3! - 2!) + (4! - 3!) + \dots + ((n+1)! - n!)]$$

$$= 1 + ((n+1)! - 1!) = (n+1)!$$

Type III PRACTICAL PROBLEMS ON PERMUTATIONS

EXAMPLE 14 *In how many ways three different rings can be worn in four fingers with at most one in each finger ?*

SOLUTION The total number of ways is same as the number of arrangements of 4 fingers, taken 3 at a time.

So, required number of ways $= {}^4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4! = 24$.

ALITER Let R_1, R_2, R_3 be three rings. Since R_1 can be put in any one of the four fingers. So, there are four ways in which R_1 can be worn. Now, R_2 can be worn in any one of the remaining three fingers in 3 ways. In the remaining 2 fingers ring R_3 can be worn in 2 ways. So, by the fundamental principle of counting the total number of ways in which three different rings can be worn in four fingers is $4 \times 3 \times 2 = 24$.

EXAMPLE 15 *Seven athletes are participating in a race. In how many ways can the first three prizes be won ?*

SOLUTION The total number of ways in which first three prizes can be won is the number of arrangements of seven different things taken 3 at a time.

So, required number of ways $= {}^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4!}{4!} = 210$

ALITER First prize can be won in seven ways. Second prize can be won by any one of the remaining six athletes in 6 ways. Now, five athletes are left. So, third prize can be won by any one of the remaining 5 athletes in 5 ways.

Hence, by the fundamental principle of counting, the required number of ways $= 7 \times 6 \times 5 = 210$.

EXAMPLE 16 *How many different signals can be made by 5 flags from 8 flags of different colours ?*

SOLUTION The total number of signals is the number of arrangements of 8 flags by taking 5 flags at a time.

Hence, required number of signals

$$= {}^8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 6720$$

EXAMPLE 17 *In how many ways can 6 persons stand in a queue ?*

SOLUTION The number of ways in which 6 persons can stand in a queue is same as the number of arrangements of 6 different things taken all at a time.

Hence, the required number of ways $= {}^6P_6 = 6! = 720$.

EXAMPLE 18 *It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?*

SOLUTION In all 9 persons are to be seated in a row and in the row of 9 positions there are exactly four even places viz second, fourth, sixth and eighth. It is given that these four even places are to be occupied by 4 women. This can be done in ${}^4P_4$ ways (ways of arranging 4 women in 4 positions). The remaining 5 positions can be filled by the 5 men in ${}^5P_5$ ways. So, by the fundamental principle of counting, the number of seating arrangements as required, is ${}^4P_4 \times {}^5P_5 = 4! \times 5! = 24 \times 120 = 2880$

EXAMPLE 19 *Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them ?*

SOLUTION The total number of ways in which three men can wear 4 coats is the number of arrangements of 4 different coats taken 3 at a time. So, three men can wear 4 coats in ${}^4P_3$ ways. Similarly, 5 waist coats and 6 caps can be worn by three men in ${}^5P_3$ and ${}^6P_3$ ways respectively. Hence, by the fundamental principle of counting, the required number of ways as desired

$$= {}^4P_3 \times {}^5P_3 \times {}^6P_3 = (4!) \times (5 \times 4 \times 3) \times (6 \times 5 \times 4) = 172800$$

EXAMPLE 20 *How many different signals can be given using any number of flags from 5 flags of different colours ?*

SOLUTION The signals can be made by using at a time one or two or three or four or five flags.

The total number of signals when r flags are used at a time from 5 flags is equal to the number of arrangements of 5, taking r at a time i.e. ${}^5P_r$. Since r can take values 1, 2, 3, 4, 5. Hence, by the fundamental principle of addition, the total number of signals

$$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$$
$$= 5 + 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1$$
$$= 5 + 20 + 60 + 120 + 120 = 325$$

EXAMPLE 21 *In an examination hall there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated ?*

SOLUTION Let the two classes be C_1 and C_2 and the four rows be R_1, R_2, R_3, R_4. There are 16 students in each class. So, there are 32 students. According to the given conditions there are two different ways in which 32 students can be seated:

	R_1	R_2	R_3	R_4
I	C_1	C_2	C_1	C_2
II	C_2	C_1	C_2	C_1

Since the seating arrangement can be completed by using any one of these two ways. So, by the fundamental principle of addition, we have

Total number of seating arrangements

= No. of arrangement in I case + No. of arrangements in II case.

Now, 16 students of class C_1 can be seated in 16 chairs in ${}^{16}P_{16} = 16!$ ways.

And, 16 students of class C_2 can be seated in 16 chairs in ${}^{16}P_{16} = 16!$ ways.

Hence, Total number of seating arrangements

$$= (16! \times 16!) + (16! \times 16!) = 2\,(16! \times 16!)$$

EXAMPLE 22 *How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed ?*

SOLUTION Every number lying between 100 and 1000 is a three digit number. Therefore, we have to find the number of permutations of five digits 1, 2, 3, 4, 5 taken three at a time. Hence, the required number of numbers is

$${}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$$

EXAMPLE 23 *How many four digit numbers are there with distinct digits ?*

SOLUTION The total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taking 4 at a time is ${}^{10}P_4$. But these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not four digit numbers.

When 0 is fixed at thousand's place, we have to arrange remaining 9 digits by taking 3 at a time. The number of such arrangements is ${}^9P_3$.

So, the total number of numbers having 0 at thousand's place $= {}^9P_3$.

Hence, the total number of four digit numbers $= {}^{10}P_4 - {}^9P_3 = 5040 - 504 = 4536$.

EXAMPLE 24 *Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Determine the number of words which have at least one letter repeated.*

SOLUTION The number of 5-letter words which can be formed from 10 letters when one or more of its letters is repeated $= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

The number of 5-letter words which can be formed when none of their letters is repeated

= Number of arrangements of 10 letters by taking 5 at a time.

$$= {}^{10}P_5 = 30240$$

Hence, the number of 5-letter words which have at least one of their letters repeated is $10^5 - 30240 = 69760$.

EXAMPLE 25 *Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.*

SOLUTION We have,

The total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time

= Number of arrangement of 4 digits, taken all at a time $= {}^4P_4 = 4! = 24$.

To find the sum of these 24 numbers, we will find the sum of digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers. Each of the digits 2, 3, 4, 5 occurs in $3!\ (= 6)$ times in the unit's place.

So, total for the digits in the unit's place in all the numbers

$$= (2 + 3 + 4 + 5) \times 3! = 84.$$

Since each of the digits 2, 3, 4, 5 occurs 3! times in any one of the remaining places.

So, the sum of the digits in the ten's, hundred's and thousand's places in all the numbers

$$= (2 + 3 + 4 + 5) \times 3! = 84.$$

Hence, the sum of all the numbers $= 84\,(10^0 + 10^1 + 10^2 + 10^3) = 93324$.

EXAMPLE 26 *In how many ways 7 pictures can be hung from 5 picture nails on a wall ?*

SOLUTION The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is same as the number of arrangements of 7 things, taking 5 at a time.

Hence, the required number $= {}^7P_5 = \frac{7!}{(7-5)!} = \frac{7!}{2!} = 2520$.

EXAMPLE 27 *Determine the number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct.*

SOLUTION The required natural numbers consist of 4 digits, 3 digits, 2 digits and one digit.

Total number of 4 digit natural numbers with distinct digits $= {}^{10}P_4 - {}^9P_3$

Total number of 3 digit natural numbers with distinct digits $= {}^{10}P_3 - {}^9P_2$

Total number of 2 digit natural numbers with distinct digits $= {}^{10}P_2 - {}^9P_1$

Total number of one digit natural numbers = 9

Hence, the required number of natural numbers

$$= ({}^{10}P_4 - {}^9P_3) + ({}^{10}P_3 - {}^9P_2) + ({}^{10}P_2 - {}^9P_1) + 9$$
$$= 9 \times 9 \times 8 \times 7 + 9 \times 9 \times 8 + 9 \times 9 + 9 = 5274.$$

EXAMPLE 28 *How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once.*

SOLUTION There are eight letters in the word 'EQUATION'. So, the total number of words is equal to the number of arrangements of these letters, taken all at a time. The number of such arrangements is $^8P_8 = 8!$. Hence, the total number of words = 8!

EXAMPLE 29 *How many 4-letter words, with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed ?*

SOLUTION There are 10 letters in the word 'LOGARITHMS'.

So, the number of 4 - letter word

= Number of arrangements of 10 letters, taken 4 at a time

$= {}^{10}P_4 = 5040$

EXERCISE 16.3

1. Evaluate each of the following :

   (i) $^8P_3$ (ii) $^{10}P_4$ (iii) $^6P_6$ (iv) $P(6, 4)$

2. If $P(5, r) = P(6, r-1)$, find r. [NCERT]
3. If $5\,P(4, n) = 6\,.\,P(5, n-1)$, find n.
4. If $P(n, 5) = 20\,.\,P(n, 3)$, find n
5. If $^nP_4 = 360$, find the value of n.
6. If $P(9, r) = 3024$, find r.
7. If $P(11, r) = P(12, r-1)$ find r.
8. If $P(n, 4) = 12\,.\,P(n, 2)$, find n.
9. If $P(n-1, 3) : P(n, 4) = 1 : 9$, find n. [NCERT]
10. If $P(2n-1, n) : P(2n+1, n-1) = 22 : 7$ find n.
11. If $P(n, 5) : P(n, 3) = 2 : 1$, find n.
12. Prove that:

    $1\,.\,P(1, 1) + 2\,.\,P(2, 2) + 3\,.\,P(3, 3) + \ldots + n\,.\,P(n, n) = P(n+1, n+1) - 1.$

13. If $P(15, r-1) : P(16, r-2) = 3 : 4$, find r.
14. If ${}^{n+5}P_{n+1} = \frac{11(n-1)}{2}\,{}^{n+3}P_n$, find n.
15. In how many ways can five children stand in a queue ?
16. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done ?
17. Four letters E, K, S and V, one in each, were purchased from a plastic warehouse. How many ordered pairs of letters, to be used as initials, can be formed from them?
18. Four books, one each in Chemistry, Physics, Biology and Mathematics, are to be arranged in a shelf. In how many ways can this be done ?
19. Find the number of different 4-letter words, with or without meanings, that can be formed from the letters of the word 'NUMBER'.
20. How many three-digit numbers are there, with distinct digits, with each digit odd?
21. How many words, with or without meaning, can be formed by using all the letters of the word 'DELHI', using each letter exactly once ?

22. How many words, with or without meaning, can be formed by using the letters of the word 'TRIANGLE' ?
23. There are two works each of 3 volumes and two works each of 2 volumes; In how many ways can the 10 books be placed on a shelf so that the volumes of the same work are not separated ?
24. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible, correct or incorrect, answers are there to this question ?
25. How many three-digit numbers are there, with no digit repeated?
26. How many 6-digit telephone numbers can be constructed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once ?
27. In how many ways can 6 boys and 5 girls be arranged for a group photograph if the girls are to sit on chairs in a row and the boys are to stand in a row behind them ?
28. If a denotes the number of permutations of $(x+2)$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x - 11$ things taken all at a time such that $a = 182\,bc$, find the value of x.
29. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated? [NCERT]
30. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7, if no digits is repeated? [NCERT]
31. Find the numbers of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, if no digit is repreated? Howmany of these will be even? [NCERT]

ANSWERS

1. (i) 336	(ii) 5040	(iii) 720	(iv) 360
2. 4	3. 3	4. 8	5. 6
6. 4	7. 9	8. 6	9. 9
10. 10	11. 5	13. 14	14. 6, 7
15. 120	16. 1260	17. 12	18. 24
19. 360	20. 60	21. 120	22. 8!
23. 3456	24. 720	25. 648	26. 1680
27. 86400	28. 12	29. 504	30. 60
31. 120, 48			

HINTS TO NCERT & SELECTED PROBLEMS

2. We have

$$P(5, r) = P(6, r-1)$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6!}{\{6-(r-1)\}!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{6 \times 5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{1}{(5-r)!} = \frac{6}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow \quad 1 = \frac{6}{(7-r)(6-r)}$$

$$\Rightarrow \quad 42 - 13r + r^2 = 6$$

$\Rightarrow \quad r^2 - 13r + 36 = 0$

$\Rightarrow \quad (r-4)(r-9) = 0$

$\Rightarrow \quad r - 4 = 0 \qquad [\because P(5, r) \text{ is meaningful for } r \leq 5 \therefore r \neq 9]$

$\Rightarrow \quad r = 4$

9. $P(n-1, 3) : P(n, 4) = 1 : 9$

$$\Rightarrow \frac{P(n-1,3)}{P(n,4)} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n!} = \frac{1}{9} \Rightarrow \frac{1}{n} = \frac{1}{9} \Rightarrow n = 9$$

15. The total no. of ways = No. of arrangements of 5 things taken all at a time = ${}^5P_5$
16. Total no. of ways = No. of arrangements of 36 things, taken two at a time = ${}^{36}P_2$
17. The total no. of ordered pairs = No. of arrangements of 4 letters, taken two at a time = ${}^4P_2$.
18. No. of ways = No. of arrangements of 4 books, taken all at a time = ${}^4P_4$.
19. Total no. of words = No. of arrangements of 6 letters, taken 4 at a time = ${}^6P_4$.
20. Required number of numbers = No. of arrangements of digits 1, 3, 5, 7, 9 by taking 3 at a time = ${}^5P_3$.
23. Let $\frac{W_{11}, W_{12}, W_{13}}{W_1}$; $\frac{W_{21}, W_{22}, W_{23}}{W_2}$; $\frac{W_{31}, W_{32}}{W_3}$; $\frac{W_{41}, W_{42}}{W_4}$ be 4 works. These 4 works can be arranged in 4! ways. Now, volumes of each work can be arranged in the following ways : $W_1 \to 3!$ ways ; $W_2 \to 3!$ ways, $W_3 \to 2!$ ways, $W_4 \to 2!$ ways. Hence, total no. of ways to arrange all books = $4! \, (3! \times 3! \times 2! \times 2!) = 3456$.
24. Each answer to the given question is an arrangement of the 6 items of column B keeping the order of items in column A fixed. Hence, the total number of answers = no. of arrangements of 6 items in column B $= {}^6P_6 = 6!$.
25. Total number of three digit numbers with distinct digits $= {}^{10}P_3 - {}^9P_2$.
26. Required number of telephone numbers $= {}^8P_4$.
27. Five girls can sit on chairs in a row in ${}^5P_5 = 5!$ ways. Also, 6 boys can stand behind them in a row in ${}^6P_6 = 6!$ ways. Hence, the total number of ways $= 5! \times 6!$.
31. The total number of 4 digit numbers formed by using the digits 1, 2, 3, 4, 5 is same as the number of arrangements of 5 digits taken 4 at a time. So, required number of numbers $= {}^5P_4 = \frac{5!}{(5-4)!} = 120$

An even number will have 2 or 4 at its unit's place. So, unit's place can be filled in 2 ways and the remaining three places (tens, hundreds and thousands) can be filled with remaining 4 digits in ${}^4P_3$ ways. Hence, total number of 4 digit even numbers formed by using the given digits is ${}^4P_3 \times 2 = 48$.

16.4 PERMUTATIONS UNDER CERTAIN CONDITIONS

In this section, we shall discuss permutations where either repetitions of items are allowed or distinction between some of the items are ignored or a particular item occurs in every arrangement etc. Such type of permutations are known as permutations under certain conditions as discussed below.

THEOREM 1 *Prove that the number of all permutations of n different objects taken r at a time, when a particular object is to be always included in each arrangement, is $r \cdot {}^{n-1}P_{r-1}$.*

PROOF Here we have to find the number of ways in which r places can be filled with n given objects such that a particular object occurs in each arrangement. Suppose the particular object is placed at the first place. Then, the remaining $(r-1)$ places can be filled with remaining $(r-1)$ objects in ${}^{n-1}P_{r-1}$ ways. Similarly, by fixing the particular object at the second, third, fourth, ..., rth places, we find that the number of permutations in each case is ${}^{n-1}P_{r-1}$. Hence, by the fundamental principle of addition, the required number of permutations $= {}^{n-1}P_{r-1} + {}^{n-1}P_{r-1} + \dots + {}^{n-1}P_{r-1} = r \cdot {}^{n-1}P_{r-1}$.

THEOREM 2 *Prove that the number of permutations of n distinct objects taken r at a time, when a particular object is never taken in each arrangement, is ${}^{n-1}P_r$.*

PROOF Since one particular object out of n given objects is never taken. So, we have to determine the number of ways in which r places can be filled with $(n-1)$ distinct objects. Clearly, the number of such arrangement is ${}^{n-1}P_r$.

THEOREM 3 *Prove that the number of permutations of n different objects taken r at a time in which two specified objects always occur together is $2!\,(r-1)\,{}^{n-2}P_{r-2}$.*

PROOF First let us leave out the two specified objects. Then the number of permutations of the remaining $(n-2)$ objects, taken $(r-2)$ at a time, is ${}^{n-2}P_{r-2}$. Now, we consider two specified objects temporarily as a single object and add it to each of these ${}^{n-2}P_{r-2}$ permutations which can be done in $(r-1)$ ways. Thus, the number of permutations becomes $(r-1)\,{}^{n-2}P_{r-2}$. But two specified things can be put together in 2! ways. Hence, the required number of permutations is $2! \cdot (r-1) \cdot {}^{n-2}P_{r-2}$.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In how many ways can the letters of the word PENCIL be arranged so that (i) N is always next to E ? (ii) N and E are always together ?*

SOLUTION (i) Let us keep EN together and consider it as one letter. Now, we have 5 letters which can be arranged in a row in ${}^5P_5 = 5! = 120$ ways. Hence, the total number of ways in which N is always next to E is 120.

(ii) Keeping E and N together and considering it as one letter, we have 5 letters which can be arranged in ${}^5P_5 = 5!$ ways. But E and N can be put together 2! ways (viz. EN, NE). Hence, the total number of ways $= 5! \times 2! = 240$.

EXAMPLE 2 *How many different words can be formed with the letters of the word EQUATION so that*

(i) the words begin with E ? (ii) the words begin with E and end with N ?

(iii) the words begin and end with a consonant ?

SOLUTION Clearly, the given word contains 8 letters out of which 5 are vowels and 3 consonants.

(i) Since all words must begin with E. So, we fix E at the first place. Now, remaining 7 letters can be arranged in ${}^7P_7 = 7!$ ways.

So, total number of words = 7 !

(ii) Since all words must begin with E and end with N. So, we fix E at the first place and N at the last place. Now, remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways. Hence, the required number of words $= {}^6P_6 = 6!$

(iii) There are 3 consonants and all words should begin and end with a consonant. So, first and last places can be filled with 3 consonants in ${}^3P_2$ ways. Now, the remaining 6 places are to be filled up with the remaining 6 letters in ${}^6P_6$ ways. Hence, the required number of words $= {}^3P_2 \times {}^6P_6 = 6 \times 720 = 4320$

EXAMPLE 3 *How many words can be formed from the letters of the word, 'TRIANGLE'? How many of these will begin with T and end with E?*

SOLUTION There are 8 letters in the word 'TRIANGLE'. The total number of words formed with these 8 letters is the number of arrangements of 8 items, taken all at a time, which is equal to ${}^8P_8 = 8! = 40320$

If we fix up T in the beginning and E at the end, then the remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways. So, the total number of words which begin with T and end with $E = 6! = 720$.

EXAMPLE 4 *How many words can be formed with the letters of the word 'ORDINATE' so that vowels occupy odd places?*

SOLUTION There are 4 vowels and 4 consonants in the word 'ORDINATE'. We have to arrange 8 letters in a row such that vowels occupy odd places. There are 4 odd places viz. 1, 3, 5, 7. Four vowels can be arranged in these 4 odd places in 4! ways. Remaining 4 even places viz. 2, 4, 6, 8 are to be occupied by the 4 consonants. This can be done in 4! ways. Hence, the total number of words in which vowels occupy odd places $= 4! \times 4! = 576$.

EXAMPLE 5 *In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?*

SOLUTION The 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. In each of these arrangements 6 places are created, shown by the cross-marks, as given below :

$$\times B \times B \times B \times B \times B \times$$

Since no two girls are to sit together, so we may arrange 3 girls in 6 places. This can be done in ${}^6P_3$ ways i.e. 3 girls can be seated in ${}^6P_3$ ways.

Hence, the total number of seating arrangements

$$= {}^5P_5 \times {}^6P_3 = 5! \times 6 \times 5 \times 4 = 14400.$$

EXAMPLE 6 *In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places?*

SOLUTION There are 5 distinct letters in the word 'DELHI'. We wish to find the total number of arrangements of these 5 letters so that vowels occupy only even places. There are two vowels E and I and 2 even places viz 2 and 4. These two vowels can be arranged in the two even places in 2! ways. The remaining three letters (D, L, H) can be arranged in 3 places (viz Ist 3rd, 5th) in 3! ways. Hence, by the fundamental principle of counting the total number of arrangements $= 3! \times 2! = 12$.

EXAMPLE 7 *How many words can be formed from the letters of the word 'DAUGHTER' so that (i) the vowels always come together? (ii) the vowels never come together?* [NCERT]

SOLUTION There are 8 letters in the word 'DAUGHTER', including 3 vowels (A, U, E) and 5 consonants (D, G, H, T, R).

(i) Considering three vowels as one letter, we have 6 letters which can be arranged in ${}^6P_6 = 6!$ ways. But corresponding each way of these arrangements, the vowels A, U, E can be put together in 3! ways.

Hence, required number of words $= 6! \times 3! = 720 \times 6 = 4320$

(ii) The total number of words formed by using all the eight letters of the word 'DAUGHTER' is ${}^8P_8 = 8! = 40320$.

So, the total number of words in which vowels are never together

= Total number of words − Number of words in which vowels are always together

$= 40320 - 4320 = 36000$

EXAMPLE 8 *In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together?*

SOLUTION The number of arrangements in which the best and the worst papers never come together can be obtained by subtracting from the total number of arrangements, the number of arrangements in which the best and worst come together.

The total number of arrangements of 9 papers $= {}^9P_9 = 9!$

Considering the best and the worst papers as one paper, we have 8 papers which can be arranged in ${}^8P_8 = 8!$ ways. But the best and worst papers can be put together in 2! ways. So, the number of permutations in which the best and the worst papers can be put together $= (2! \times 8!)$.

Hence, the number of ways in which the best and the worst papers never come together $= 9! - 2! \times 8! = 9 \times 8! - 2 \times 8! = 7 \times 8! = 282240$.

EXAMPLE 9 *In how many ways can 5 children be arranged in a row such that*

(i) *two of them, Ram and Shyam, are always together?*

(ii) *two of them, Ram and Shyam, are never together?*

SOLUTION There are five children including Ram and Shyam.

(i) Considering Ram and Shyam as one child, there are four children. They can be arranged in a row in 4! ways. But Ram and Shyam can be arranged together in 2! ways. Hence, the required number of arrangements $= 4! \times 2! = 48$.

(ii) Total number of arrangements of 5 children in a row $= 5! = 120$.

∴ Total number of arrangements in which Ram and Shyam are never together

= Total number of arrangements − Number of arrangements in which Ram and Shyam are together

$= 120 - 48 = 72$.

EXAMPLE 10 *When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the Principal, how many arrangements are possible?*

SOLUTION Since the middle seat of the front row is reserved for the Principal, the remaining 6 teachers can be arranged in the front row in ${}^6P_6 = 6!$ ways.

The two corners of the second row are reserved for the two tallest students. They can occupy these two places in 2! ways. The remaining 18 seats may be occupied by the remaining 18 students in 18! ways.

Hence, by the fundamental principle of counting, the total number of arrangements

$$= 6! \times (18! \times 2!) = 18! \times 1440.$$

EXAMPLE 11 *How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?*

SOLUTION We have to determine the total number of even numbers formed by using the given condition. So, at unit's place we can use one of the digits 0, 2, 4, 6, 8. If 5 is at ten's place then, as per the given condition, 7 should be at unit's place. In such a case the number will not be an even number. So, 5 cannot be at ten's and one's places. Hence, 5 can be only at hundred's place. Now two cases arise.

CASE I When 5 is at hundred's place

If 5 is at hundred's place, then 7 will be at ten's place. So, unit's place can be filled in 5 ways by using the digits 0, 2, 4, 6, 8.

So, total number of even numbers $= 1 \times 1 \times 5 = 5$.

CASE II When 5 is not at hundred's place

Now, hundred's place can be filled in 8 ways (0 and 5 cannot be used at hundred's place). In ten's place we can use any one of the ten digits except 5. So, ten's place can be filled in 9 ways. At unit's place we have to use one of the even digits 0, 2, 4, 6, 8. So, units place can be filled in 5 ways.

So, total number of even numbers $= 8 \times 9 \times 5 = 360$

Hence, the total number of required even numbers $= 360 + 5 = 365$.

EXAMPLE 12 *A code word is to consist of two distinct English alphabets followed by two distinct numbers from 1 to 9. For example, CA 23 is a code word. How many such code words are there? How many of them end with an even integer ?*

SOLUTION There are 26 English alphabets. So, first two places in the code word can be filled in ${}^{26}P_2$ ways. In last two places we have to use two distinct numbers from 1 to 9. So, last two places can be filled in ${}^{9}P_2$ ways. Hence, by the fundamental principle of counting, the total number of code words

$$= {}^{26}P_2 \times {}^{9}P_2 = 650 \times 72 = 46800.$$

Number of code words ending with an even integer.

In this case, the code word can have any of the numbers 2, 4, 6, 8 at the extreme right position. So, the extreme right position can be filled in 4 ways. Now, next left position can be filled with any one of the remaining 8 digits in 8 ways and the two extreme left positions can be filled by two English alphabets in ${}^{26}P_2$ ways.

Hence, the total number of code words which end with an even integer

$$= 4 \times 8 \times {}^{26}P_2 = 4 \times 8 \times 650 = 20800.$$

EXAMPLE 13 *The Principal wants to arrange 5 students on the platform such that the boy 'SALIM' occupies the second position and such that the girl, 'SITA' is always adjacent to the girl 'RITA'. How many such arrangements are possible ?*

SOLUTION Since SALIM occupies the second position and the two girls RITA and SITA are always adjacent to each other. So, none of these two girls can occupy the first seat. Thus, first seat can be occupied by any one of the remaining two students in 2 ways. Second seat can be occupied by SALIM in only one way.

Now, in the remaining three seats SITA and RITA can be seated in the following four ways:

	I	II	III	IV	V
1.	×	SALIM	SITA	RITA	×
2.	×	SALIM	RITA	SITA	×
3.	×	SALIM	×	SITA	RITA
4.	×	SALIM	×	RITA	SITA

Now, only one seat is left which can be occupied by the 5th student in one way.

Hence, the number of required type of arrangements $= 2 \times 4 \times 1 = 8$.

EXAMPLE 14 *How many numbers between 400 and 1000 can be formed with the digits 0, 2, 3, 4, 5, 6 if no digit is repeated in the same number ?*

SOLUTION Number between 400 and 1000 consist of three digits with digit at hundred's place greater than or equal to 4. Hundred's place can be filled, by using the digits 4, 5, 6 in 3 ways. Now, ten's and unit's places can be filled by the remaining 5 digits in ${}^{5}P_2$ ways.

Hence, the required number of numbers $= 3 \times {}^{5}P_2 = 3 \times \frac{5!}{3!} = 3 \times 20 = 60$.

EXAMPLE 15 *How many four digit numbers divisible by 4 can be made with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed ?*

SOLUTION Recall that a number is divisible by 4 if the number formed by the last two digits is divisible by 4. The digits at unit's and ten's places can be arranged as follows:

Th	H	T	O
×	×	1	2
×	×	2	4
×	×	3	2
×	×	5	2

Now, corresponding each such way the remaining three digits at thousand's and hundred's places can be arranged in ${}^{3}P_2$ ways.

Hence, the required number of numbers $= {}^{3}P_2 \times 4 = 3! \times 4 = 24$.

EXAMPLE 16 *Find the number of ways in which 5 boys and 5 girls be seated in a row so that*

(i) *No two girls may sit together.*

(ii) *All the girls sit together and all the boys sit together.*

(iii) *All the girls are never together.*

SOLUTION

(i) 5 boys can be seated in a row in ${}^{5}P_5 = 5!$ ways. Now, in the 6 gaps 5 girls can be arranged in ${}^{6}P_5$ ways.

Hence, the number of ways in which no two girls sit together

$$= 5! \times {}^{6}P_5 = 5! \times 6!$$

(ii) The two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements $= 2!\,(5! \times 5!) = 2\,(5!)^2$.

(iii) The total number of ways in which all the girls are never together

= Total number of arrangements − Total number of arrangements in which all the girls are always together

$$= 10! - 5! \times 6!$$

EXAMPLE 17 *Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line.*

SOLUTION 5 boys can be arranged in a line in ${}^{5}P_5 = 5!$ ways. Since the boys and girls are alternating. So, corresponding each of the 5! ways of arrangements of 5 boys we obtain 5 places marked by cross as shown below:

(i) $B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times$ (ii) $\times B_1 \times B_2 \times B_3 \times B_4 \times B_5$

Clearly, 5 girls can be arranged in 5 places marked by cross in $(5! + 5!)$ ways.

Hence, the total number of ways of making the line $= 5! \times (5! + 5!) = 2(5!)^2$

EXAMPLE 18 *In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats?*

SOLUTION We have : Total number of persons = 3 girls + 9 boys = 12.

Total number of numbered seats $= 2 \times 3 + 4 \times 2 = 14$

So, total number of ways in which 12 persons can be seated on 14 seats = Number of arrangements or 14 seats by taking 12 at a time $= {}^{14}P_{12}$.

Three girls can be seated together in a back row on adjacent seats in the following ways

1, 2, 3 or 2, 3, 4 of first van

and 1, 2, 3 or 2, 3, 4 of second one.

In each way the three girls can interchange among themselves in 3! ways. So, the total number of ways in which three girls can be seated together in a back row on adjacent seats $= 4 \times 3!$

Now, 9 boys are to be seated on remaining 11 seats, which can be done in ${}^{11}P_9$ ways.

Hence, by the fundamental principle of counting, the total number of seating arrangements is ${}^{11}P_9 \times 4 \times 3!$.

EXAMPLE 19 *A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular and two on the other side. In how many ways can they be seated ?*

SOLUTION Let the two sides be A and B. Assume that four persons wish to sit on side A. Four persons who wish to sit on side A can be accommodated on eight chairs in ${}^8P_4$ ways and two persons who wish to sit on side B can be accommodated on 8 chairs in ${}^8P_2$ ways. Now, 10 persons are left, who can sit on 10 chairs on both the sides of the table in 10! ways.

Hence, the total number of ways in which 16 persons can be seated

$$= {}^8P_4 \times {}^8P_2 \times 10!$$

EXAMPLE 20 *In a class of 10 students there are 3 girls A, B, C. In how many different ways can they be arranged in a row such that no two of the three girls are consecutive.*

SOLUTION There are 7 boys and 3 girls. Seven boys can be arranged in a row in ${}^7P_7 = 7!$ ways. Now, we have 8 places in which we can arrange 3 girls in ${}^8P_3$ ways.

Hence, by the fundamental principle of counting, the number of arrangements

$$= 7! \times {}^8P_3 = 7! \times 336.$$

EXERCISE 16.4

1. In how many ways can the letters of the word 'FAILURE' be arranged so that the consonants may occupy only odd positions?
2. In how many ways can the letters of the word 'STRANGE' be arranged so that
   (i) the vowels come together ?
   (ii) the vowels never come together? and
   (iii) the vowels occupy only the odd places?
3. How many words can be formed from the letters of the word 'SUNDAY'? How many of these begin with D?
4. How many words can be formed out of the letters of the word, 'ORIENTAL', so that the vowels always occupy the odd places ?

5. How many different words can be formed with the letters of word 'SUNDAY'? How many of the words begin with N? How many begin with N and end in Y?
6. How many different words can be formed from the letters of the word 'GANESHPURI' ? In how many of these words:
   (i) the letter G always occupies the first place?
   (ii) the letters P and I respectively occupy first and last place?
   (iii) the vowels are always together?
   (iv) the vowels always occupy even places?
7. How many permutations can be formed by the letters of the word, 'VOWELS', when
   (i) there is no restriction on letters?
   (ii) each word begins with E?
   (iii) each word begins with O and ends with L?
   (iv) all vowels come together?
   (v) all consonants come together?
8. How many words can be formed out of the letters of the word 'ARTICLE', so that vowels occupy even places ?
9. In how many ways can a lawn tennis mixed double be made up from seven married couples if no husband and wife play in the same set ?
10. m men and n women are to be seated in a row so that no two women sit together. If $m > n$ then show that the number of ways in which they can be seated as
   $$\frac{m!\,(m+1)!}{(m-n+1)!}$$
11. How many words (with or without dictionary meaning) can be made from the letters in the word MONDAY, assuming that no letter is repeated, if
   (i) 4 letters are used at a time?
   (ii) all letters are used at a time.
   (iii) all letters are used but first is vowel.
12. How many three letter words can be made using the letters of the word 'ORIENTAL'?

ANSWERS

1. 576	2. (i) 1440	(ii) 3600	(iii) 1440
3. 720, 120	4. 576	5. 720, 120, 24	
6. 10!	(i) 9!	(ii) 8!	(iii) 7! × 4!
(iv) 5! × 6!	7. (i) 720	(ii) 120	(iii) 24
(iv) 240	(v) 144		
8. 144	9. 840	11. (i) 360 (ii) 720 (iii) 240	12. 336

16.5 PERMUTATIONS OF OBJECTS NOT ALL DISTINCT

So far we were discusing permutations of distinct objects (things) by taking some or all at a time. In this article we intend to discuss the permutations of a given number of objects when objects are not all different. For example, the number of arrangements of the letters of the word MISSISSIPPI, the number of six digit numbers formed by using the digits 1, 1, 2, 3, 3, 4 etc. The following theorem is very helpful to determine the number of such arrangements.

THEOREM *The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is*

$$\frac{n!}{p!\,q!}$$

PROOF Let the required number of permutations be x. Consider one of these x permutations.

Now, replace p alike things in this permutation by p distinct things which are also different from others. These p different things may be permuted among themselves in $p!$ ways without changing the positions of other things. Similarly, if we replace q alike things by q distinct things, which are also different from others, then they can be permuted among themselves in $q!$ ways.

Thus, if both the replacements are done simultaneously, then we find that each one of the x permutations give rise to $p! \times q!$ permutations. Therefore x permutations give rise to $x \times p! \times q!$ permutations. Now, each of these $x \times p! \times q!$ permutations, is a permutation of n different things, taken all at a time.

$\therefore \quad x \times p! \times q! =$ No. of permutations of n different things taken all at a time $= n!$

Hence, $\quad x = \dfrac{n!}{p!\,q!}$

REMARK 1 *The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ... ; p_r are alike of rth kind such that $p_1 + p_2 + ... + p_r = n$, is*

$$\frac{n!}{p_1!\,p_2!\,p_3! \dots p_r!}$$

REMARK 2 *The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is*

$$\frac{n!}{p!\,q!}$$

REMARK 3 *Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \dots, p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is*

$$\frac{(p_1 + p_2 + \dots + p_r)!}{p_1!\,p_2!\,p_3! \dots p_r!}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *How many different words can be formed with the letters of the word 'MISSISSIPPI'? In how many of these permutations four I's do not come together?* [NCERT]

SOLUTION There are 11 letters in the given word, of which 4 are S's, 4 are I's and 2 are P's. So, total number of words is the number of arrangements of 11 things, of which 4 are similar of one kind, 4 are similar of second kind and 2 are similar of third kind i.e.

$$\frac{11!}{4!\,4!\,2!}$$

Hence, the total number of words $= \dfrac{11!}{4!\,4!\,2!} = 34650.$

Considering 4 I's as one letter, we have 8 letters of which 4 are S's and 2 are P's. These 8 letters can be arranged in $\dfrac{8!}{4!\,2!}$ ways.

Number of words in which 4 I's come together $= \dfrac{8!}{4!\,2!} = 840$

Hence, number of words in which 4 I's do not come together $= 34650 - 840 = 33810.$

EXAMPLE 2 *How many permutations of the letters of the word 'APPLE' are there?*

SOLUTION Here there are 5 letters, two of which are of the same kind. The others are each of its own kind. So, the required number of permutations is

$$\frac{5!}{2!\,1!\,1!\,1!} = \frac{120}{2} = 60.$$

EXAMPLE 3 *How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice?*

SOLUTION We are given 8 letters viz. A, A, A, B, B, C, C, C. Clearly, there are 8 letters of which three are of one kind, two are of second kind and three are of third kind. So, the total number of permutations is $\dfrac{8!}{3!\,2!\,3!} = 560.$

Hence, the requisite number of words $= 560.$

EXAMPLE 4 *Find the number of different permutations of the letters of the word BANANA?*

SOLUTION Clearly, there are six letters in the word 'BANANA' of which three are alike of one kind (3 A's), two are alike of second kind (2 N's) and one of its own kind.

$\therefore \quad$ Total number of their permutations $= \dfrac{6!}{3!\,2!\,1!} = 60$

Hence, the requisite number of words $= 60$

EXAMPLE 5 (i) *How many different words can be formed with the letters of the word HARYANA?*

(ii) *How many of these begin with H and end with N?*

(iii) *In how many of these H and N are together?*

SOLUTION

(i) There are 7 letters in the word 'HARYANA' of which 3 are A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{7!}{3!\,1!\,1!\,1!\,1!} = \dfrac{7!}{3!} = 840.$

(ii) After fixing H in first place and N in last place, we have 5 letters out of which three are alike i.e. A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{5!}{3!} = 20.$

(iii) Considering H and N together we have $7 - 2 + 1 = 6$ letters out of which three are alike i.e. A's and others are each of its own kind. These six letters can be arranged in $\dfrac{6!}{3!}$ ways. But H and N can be arranged amongst themselves in 2! ways.

Hence, the requisite number of words $= \dfrac{6!}{3!} \times 2! = 120 \times 2 = 240.$

EXAMPLE 6 *How many different words can be formed by using all the letters of the word 'ALLAHABAD'?* [NCERT]

(i) *In how many of them vowels occupy the even positions?*

(ii) *In how many of them both L do not come together?*

SOLUTION There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So, the requisite number of words $= \dfrac{9!}{4!\,2!} = 7560.$

(i) There are 4 vowels and all are alike i.e. 4 A's. Also, there are 4 even places viz 2nd, 4th, 6th and 8th. So, these 4 even places can be occupied by 4 vowels in

$\frac{4!}{4!} = 1$ way. Now, we are left with 5 places in which 5 letters, of which two are alike (2 L's) and other distinct, can be arranged in $\frac{5!}{2!}$ ways.

Hence, the total number of words in which vowels occupy the even places

$$= \frac{5!}{2!} \times \frac{4!}{4!} = \frac{5!}{2!} = 60.$$

(ii) Considering both L together and treating them as one letter we have 8 letters out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\frac{8!}{4!}$ ways.

So, the number of words in which both L come together $= \frac{8!}{4!} = 1680$.

Hence, the number of words in which both L do not come together
= Total no. of words – No. of words in which both L come together
$= 7560 - 1680 = 5880$.

EXAMPLE 7 *Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements,* **[NCERT]**

(i) *do the words start with P?*
(ii) *do all the vowels always occur together?*
(iii) *do all the vowels never occur together?*
(iv) *do the words begin with I and end in P?*

SOLUTION In the word 'INDEPENDENCE' there are 12 letters of which 3 are N's, 4 are E's and 2 are D's. Therefore,

$$\text{Total number of arrangements} = \frac{12!}{3!\,4!\,2!} = 1663200$$

(i) After fixing the letter P at the extreme left position, there are 11 letters consisting of 3 N's, 4E's and 2D's. These 11 letters can be arranged in $\frac{11!}{3!\,4!\,2!} = 138600$

$\therefore$ Number of words beginning with $P = \frac{11!}{3!\,4!\,2!} = 138600$

(ii) There are 5 vowels in the given word of which 4 are E's and one I. These vowels can be put together in $\frac{5!}{4!\,1!}$ ways. Considering these 5 vowels as one letter there are 8 letters (taking 7 remaining letters) which can be arranged in $\frac{8!}{3!\,2!}$ ways (as there are 3 N's and 2D's). Since corresponding to each arrangement of 5 vowels there are $\frac{8!}{3!\,2!}$ ways of arranging remaining 7 letters and one letter formed by 5 vowels.

Hence, by fundamental principle of multiplication, the required number of arrangements is $\frac{8!}{3!\,2!} \times \frac{5!}{4!\,1!} = 16800$

(iii) The required number of arrangements
= The total number of arrangements – The number of arrangements in which all the vowels occur together
$= 1663200 - 16800 = 1646400$

(iv) Let us fix I at the extreme left end and P at the extreme right end. Now, we are left with 10 letters of which 3 are N's, 4 are E's and 2 are D's. These ten letters can be arranged in $\frac{10!}{4!\,3!\,2!}$ ways.

Hence, required number of arrangements $= \frac{10!}{4!\,3!\,2!} = 12600$

EXAMPLE 8 *In how many ways can the letters of the word PERMUTATIONS be arranged if (i) the words start with P and with S (ii) vowels are all together.*

SOLUTION (i) There are 12 letters in the given word of which 2 are T's and the remaining are distinct. Remaining 10 letters between P and S can be arranged in $\frac{10!}{2!}$ ways.

$\therefore$ Total number of words starting with P and ending in S is $\frac{10!}{2!} = 1814400$

(ii) There are 5 vowels in the given word. These vowels can be put together in 5! ways. Considering these 5 vowels as one letter, we have 8 letters (7 remaining letters and one letter formed by 5 vowels) of which 2 are T's. These 8 letters can be arranged in $\frac{8!}{2!}$ ways.

Hence, by the fundamental principle of multiplication, required number of words is $5! \times \frac{8!}{2!} = 2419200$

EXAMPLE 9 *How many arrangements can be made with the letters of the word 'MATHEMATICS'? In how many of them vowels are together ?*

SOLUTION There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct. So,

$$\text{Required number of arrangements} = \frac{11!}{2! \times 2! \times 2!} = 4989600$$

There are 4 vowels viz. A, E, A, I. Considering these four vowels as one letter we have 8 letters (M, T, H, M, T, C, S and one letter obtained by combining all vowels), out of which M occurs twice, T occurs twice and the rest all different. These 8 letters can be arranged in $\frac{8!}{2! \times 2!}$ ways.

But the four vowels (A, E, A, I) can be put together in $\frac{4!}{2!}$ ways.

Hence, the total number of arrangements in which vowels are always together

$$= \frac{8!}{2! \times 2!} \times \frac{4!}{2!} = 10080 \times 12 = 120960.$$

EXAMPLE 10 *If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word?* **[NCERT]**

SOLUTION In dictionary the words at each stage are arranged in alphabetical order. Starting with the letter A, and arranging the other four letters GAIN, we obtain $4! = 24$ words.

Thus, there are 24 words which start with A. These are the first 24 words.

Then, starting with G, and arranging the other four letters A, A, I, N in different ways, we obtain $\frac{4!}{2!} = \frac{24}{2} = 12$ words.

Thus, there are 12 words, which start with G.

Now, we start with I. The remaining 4 letters A, G, A, N can be arranged in $\frac{4!}{2!} = 12$ ways.

So, there are 12 words, which start with I.

Thus, we have so far constructed 48 words.

The 49th word is NAAGI and hence the 50th word is NAAIG.

EXAMPLE 11 *The letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.*

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time i.e. A will occur 5! times. Similarly, D, M, N, O will occur in the first place the same number of times.

$\therefore$ Number of words starting with A = 5! = 120
Number of words starting with D = 5! = 120
Number of words starting with M = 5! = 120
Number of words starting with N = 5! = 120
Number of words starting with O = 5! = 120

Number of words beginning with R is 5!, but one of these words is the word RANDOM. So, we first find the number of words beginning with RAD and RAM.
No. of words starting with RAD = 3! = 6
No. of words starting with RAM = 3! = 6
Now, the words beginning with 'RAN' must follow.
There are 3! words beginning with RAN. One of these words is the word RANDOM itself.
The first word beginning with RAN is the word RANDMO and the next word is RANDOM.

$\therefore$ Rank of RANDOM $= 5 \times 120 + 2 \times 6 + 2 = 614$.

EXAMPLE 12 *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?*

SOLUTION Any number of greater than a million will contain all the seven digits. Now, we have to arrange these seven digits, out of which 2 occur twice, 3 occurs twice and the rest are distinct.

The number of such arrangements $= \dfrac{7!}{2! \times 3!} = 420$.

These arrangements also include those numbers which contain 0 at the million's place. Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct. These 6 digits can be arranged in $\dfrac{6!}{2! \times 3!} = 60$ ways.

Hence, the number of required numbers = 420 − 60 = 360.

EXAMPLE 13 *If the different permutations of the word, 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E?* [NCERT]

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we have to find the total number of words starting with A, because the very next word will start with E.
For finding the number of words starting with A, we have to find the number of arrangements of the remaining 10 letters, EXMINATION, of which there are 2 I's, 2 N's and the others each of its own kind.

The number of such arrangements $= \dfrac{10!}{2!\,2!} = 907200$.

Hence, the required number of items = 907200.

EXAMPLE 14 *There are six periods in each working day of a school. In how many ways can one arrange 5 subjects such that each subject is allowed at least one period ?*

SOLUTION Since each subject is allowed at least one period.
So we first select one subject for the left out period. This can be done in $^5C_1$ ways.

Now, six subject can be arranged in $\dfrac{6!}{2!}$ ways.

Hence, the total number of arrangements $= {}^5C_1 \times \dfrac{6!}{2!} = 1800$

EXERCISE 16.5

1. Find the number of words formed by permuting all the letters of the following words:
   (i) INDEPENDENCE (ii) INTERMEDIATE (iii) ARRANGE
   (iv) INDIA (v) PAKISTAN (vi) RUSSIA
   (vii) SERIES (viii) EXERCISES (ix) CONSTANTINOPLE
2. In how many ways can the letters of the word 'ALGEBRA' be arranged without changing the relative order of the vowels and consonants ?
3. How many words can be formed with the letters of the word 'UNIVERSITY', the vowels remaining together ?
4. Find the total number of arrangements of the letters in the expression $a^3 b^2 c^4$ when written at full length.
5. How many words can be formed with the letters of the word 'PARALLEL' so that all L's do not come together ?
6. How many words can be formed by arranging the letters of the word 'MUMBAI' so that all M's come together ?
7. If the letters of the word 'LATE' be permuted and the words so formed be arranged as in a dictionary, find the rank of the word LATE.
8. If the letters of the word 'MOTHER' are written in all possible orders and these words are written out as in a dictionary, find the rank of the word 'MOTHER'.
9. If the permutations of a, b, c, d, e taken all together be written down in alphabetical order as in dictionary and numbered, find the rank of the permutation debac.
10. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places ?
11. Find the total number of ways in which six '+' and four '−' signs can be arranged in a line such that no two '−' signs occur together.
12. How many different signals can be made from 4 red, 2 white and 3 green flags by arranging all of them vertically on a flagstaff ?
13. How many number of four digits can be formed with the digits 1, 3, 3, 0 ?
14. In how many ways can the letters of the word 'ARRANGE' be arranged so that the two R's are never together ?
15. How many different numbers, greater than 50000 can be formed with the digits 0, 1, 1, 5, 9.
16. How many words can be formed from the letters of the word 'SERIES' which start with S and end with S ?
17. How many permutations of the letters of the word 'MADHUBANI' do not begin with M but end with I ?
18. Find the number of numbers, greater than a million, that can be formed with the digits 2, 3, 0, 3, 4, 2, 3.
19. In how many ways can the letters of the word "INTERMEDIATE" be arranged so that:
   (i) the vowels always occupy even places ?
   (ii) the relative order of vowels and consonants do not alter ?

20. There are three copies each of 4 different books. In how many ways can they be arranged in a shelf ?
21. The letters of the word 'ZENITH' are written in all possible orders. How many words are possible if all these words are written out as in a dictionary ? What is the rank of the word 'ZENITH' ?
22. How many different arrangements can be made by using all the letters in the word 'MATHEMATICS'. How many of them begin with C? How many of them begin with T?
23. The letters of the word 'SURITI' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'SURITI'.
24. A biologist studying the genetic code is interested to know the number of possible arrangements of 12 molecules in a chain. The chain contains 4 different molecules represented by the initials *A* (for Adenine), *C* (for Cytosine), *G* (for Guanine) and *T* (for Thymine) and 3 molecules of each kind. How many different such arrangements are possible?
25. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable ? [NCERT]
26. How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4? [NCERT]
27. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together? [NCERT]
28. Find the total number of permutations of the letters of the word 'INSTITUTE'. [NCERT]

ANSWERS

1. (i) 1663200	(ii) 19958400	(iii) 1260
(iv) 60	(v) 20160	(vi) 360
(vii) 180	(viii) 30240	(ix) $\frac{14!}{24}$
2. 72	3. 60480	4. 1260
5. 3000	6. 120	7. 14
8. 309	9. 93	10. 18
11. 35	12. 1260	13. 9
14. 900	15. 24	16. 12
17. 17640	18. 360	
19. (i) 21600	(ii) 21600	20. $12!/(3!)^4$
21. 616	22. $\frac{11!}{2!\,2!\,2!}, \frac{10!}{2!\,2!\,2!}, \frac{10!}{2!\,2!}$	23. 236
24. 369600	25. 1260	26. 360
27. 151200	28. $\frac{9!}{2!\,3!}$	

HINTS TO NCERT & SELECTED PROBLEMS

2. The consonants can be arranged among themselves in 4! ways and the vowels among themselves in $\frac{3!}{2!}$ ways.

Hence, the required number of arrangements $= 4! \times \frac{3!}{2!} = 72$

4. There are 3 *a*'s, 2 *b*'s and 4 *c*'s. So,

the total number of arrangements $= \frac{9!}{3!\,2!\,4!} = 1260$.

10. There are 4 odd digits 1, 1, 3, 3 and 4 odd places. So, odd digits can be arranged in odd places in $\frac{4!}{2!\,2!}$ ways. The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\frac{3!}{2!}$ ways.

Hence, the requisite number of numbers $= \frac{4!}{2!\,2!} \times \frac{3!}{2!} = 18$.

11. Six '+' signs can be arranged in a row in $\frac{6!}{6!} = 1$ way. Now, we are left with seven places in which four different things can be arranged in ${}^7P_4$ ways but as all the four '–' signs are identical, therefore, four '–' signs can be arranged in

$\frac{{}^7P_4}{4!} = 35$ ways.

Hence, the required number of ways $= 1 \times 35 = 35$.

12. We have to arrange 9 flags, out of which 4 are of one kind, 2 are of another kind and 3 are of third kind.

So, total number of signals $= \frac{9!}{4!\,2!\,3!}$.

13. Required number of numbers $= \frac{4!}{2!} - \frac{3!}{2!}$

15. Numbers greater than 50000 will have either 5 or 9 in the first place and will consist of 5 digits.

Number of numbers of with digit 5 at first place $= \frac{4!}{2!}$

Number of numbers with digit 9 at first place $= \frac{4!}{2!}$

Hence, the required number of numbers $= \frac{4!}{2!} + \frac{4!}{2!} = 24$.

25. Required number of ways $= \frac{(4+3+2)!}{4!\,3!\,2!} = \frac{9!}{4!\,3!\,2!} = 1260$

26. Number of numbers greater than 1000000 that can be formed by using the digits 1, 2, 0, 2, 4, 2, 4.

= Number of numbers formed by given digits – Number of numbers having 0 as left most digit

$= \frac{7!}{3!\,2!} - \frac{6!}{3!\,2!} = \frac{7! - 6!}{3!\,2!} = \frac{6 \times 6!}{3!\,2!} = 360$

27. Considering all S as one letter there are 10 letters containing 3*A*'s, 2*I*'s, 2*N*'s, 1*T*, 1*O* which can be arranged in $\frac{10!}{3!\,2!\,2!} = 151200$ ways.

28. There are 9 letters in the word INSTITUTE containing 2*I*'s, 3*T*'s, 1*N*, 1*S*, 1*U* and 1*E*. These letters can be arranged in $\frac{9!}{2!\,3!} = 21040$ ways.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. In how many ways can 4 letters be posted in 5 letter boxes?
2. Write the number of 5 digit numbers that can be formed using digits 0, 1 and 2.
3. In how many ways 4 women draw water from 4 taps, if no tap remains unused?
4. Write the total number of possible outcomes in a throw of 3 dice in which at least one of the dice shows an even number.
5. Write the number of arrangements of the letters of the word BANANA in which two N's come together.
6. If the letters of the word 'LATE' be permuted and the words so formed be arranged as in a dictionary, what is the rank of 'LATE'?
7. Write the number of words that can be formed out of the letters of the word 'COMMITTEE'.
8. Write the number of all possible words that can be formed using the letters of the word 'MATHEMATICS'.
9. Write the number of ways in which 6 men and 5 women can dine at a round table if no two women sit together.
10. Write the number of ways in which 5 boys and 3 girls can be seated in a row so that each girl is between 2 boys.
11. Write the remainder obtained when $1! + 2! + 3! + \ldots + 200!$ is divided by 14.
12. Write the number of numbers that can be formed using all for digits 1, 2, 3, 4.
13. Write the number of ways in which 7 men and 7 women can sit on a round table such that no two women sit together.

ANSWERS

1. 5^4 2. 2×3^4 3. 4! 4. 189 5. 20 6. 14 7. $\frac{9!}{(2!)^3}$
8. $\frac{11!}{2!\,2!\,2!}$ 9. $6! \times 5!$ 10. 2880 11. 5 12. 24 13. $7! \times 6!$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The number of permutations of n different things taking r at a time when 3 particular things are to be included is
   (a) ${}^{n-3}P_{r-3}$ (b) ${}^{n-3}P_r$ (c) ${}^{n}P_{r-3}$ (d) $r!\,{}^{n-3}C_{r-3}$
2. Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to
   (a) 60 (b) 120 (c) 7200 (d) none of these.
3. The number of words that can be formed out of the letters of the word 'COMMITTEE' is
   (a) $\frac{9!}{(2!)^3}$ (b) $\frac{9!}{(2!)^2}$ (c) $\frac{9!}{2!}$ (d) 9!
4. How many numbers greater than 10 lacs be formed from 2, 3, 0, 3, 4, 2, 3?
   (a) 420 (b) 360 (c) 400 (d) 300

5. The number of different signals which can be given from 6 flags of different colours taking one or more at a time, is
   (a) 1958 (b) 1956 (c) 16 (d) 64
6. The number of words from the letters of the word 'BHARAT' in which B and H will never come together, is
   (a) 360 (b) 240 (c) 120 (d) none of these.
7. The number of six letter words that can be formed using the letters of the word "ASSIST" in which S's alternate with other letters is
   (a) 12 (b) 24 (c) 18 (d) none of these.
8. The number of arrangements of the word "DELHI" in which E precedes I is
   (a) 30 (b) 60 (c) 120 (d) 59
9. The number of ways in which the letters of the word 'CONSTANT' can be arranged without changing the relative positions of the vowels and consonants is
   (a) 360 (b) 256 (c) 444 (d) none of these.
10. The number of ways to arrange the letters of the word CHEESE are
    (a) 120 (b) 240 (c) 720 (d) 6
11. Number of all four digit numbers having different digits formed of the digits 1, 2, 3, 4 and 5 and divisible by 4 is
    (a) 24 (b) 30 (c) 125 (d) 100
12. If the letters of the word KRISNA are arranged in all possible ways and these words are written out as in a dictionary, then the rank of the word KRISNA is
    (a) 324 (b) 341 (c) 359 (d) none of these
13. If in a group of n distinct objects, the number of arrangements of 4 objects is 12 times the number of arrangements of 2 objects, then the number of objects is
    (a) 10 (b) 8 (c) 6 (d) none of these.
14. The number of ways in which 6 men can be arranged in a row so that three particular men are consecutive, is
    (a) $4! \times 3!$ (b) 4! (c) $3! \times 3!$ (d) none of these.
15. A 5-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is
    (a) 216 (b) 600 (c) 240 (d) 3125
16. The product of r consecutive positive integers is divisible by
    (a) $r!$ (b) $(r-1)!$ (c) $(r+1)!$ (d) none of these.
17. If ${}^{k+5}P_{k+1} = \frac{11(k-1)}{2}\,{}^{k+3}P_k$, then the values of k are
    (a) 7 and 11 (b) 6 and 7 (c) 2 and 11 (d) 2 and 6
18. In a packet there are m different books, n different pens and p different pencils. The number of selections of at least one article of each type from the packet is
    (a) $2^{m+n+p} - 1$ (b) $(m+1)(n+1)(p+1) - 1$
    (c) 2^{m+n+p} (d) none of these.
19. The number of words that can be made by re-arranging the letters of the word APURBA so that vowels and consonants are alternate is
    (a) 18 (b) 35 (c) 36 (d) none of these.

20. The number of different ways in which 8 persons can stand in a row so that between two particular persons A and B there are always two persons, is

(a) $60 \times 5!$ (b) $15 \times 4! \times 5!$ (c) $4! \times 5!$ (d) none of these

21. The number of ways in which the letters of the word ARTICLE can be arranged so that even places are always occupied by consonants is

(a) 576 (b) $^4C_3 \times 4!$ (c) $2 \times 4!$ (d) none of these

22. In a room there are 12 bulbs of the same wattage, each having a separate switch. The number of ways to light the room with different amounts of illumination is

(a) $12^2 - 1$ (b) 2^{12} (c) $2^{12} - 1$ (d) none of these

23. The number of arrangements of the letters of the word BHARAT taking 3 at a time is

(a) 72 (b) 120 (c) 14 (d) none of these.

24. The number of five-digit telephone numbers having at least one of their digits repeated is

(a) 90000 (b) 100000 (c) 30240 (d) 69760

25. The number of words that can be formed out of the letters of the word "ARTICLE" so that vowels occupy even places is

(a) 574 (b) 36 (c) 754 (d) 144

ANSWERS

1. (d)	2. (c)	3. (a)	4. (b)	5. (b)	6. (b)	7. (a)	8. (b)
9. (a)	10. (a)	11. (a)	12. (a)	13. (c)	14. (a)	15. (a)	16. (a)
17. (b)	18. (a)	19. (c)	20. (a)	21. (a)	22. (c)	23. (a)	24. (d)
25. (d)							

SUMMARY

1. The continued product of first n natural numbers is called the "n factorial" and is denoted by $\underline{|n}$ or $n!$.
   Thus, $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$
   Factorials of proper fractions and negative integers are not defined.
2. $\frac{(2n)!}{n!} = 1 \cdot 3 \cdot 5 \ldots (2n-1)\, 2^n$
3. $n! + 1$ is not divisible by any natural number between 2 and n.
4. Let p be a prime number and n be a natural number, if $E_p(n)$ denotes the exponent of p in n, then
   $$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots + \left[\frac{n}{p^s}\right]$$
   where s is the largest positive integer such that $p^s \le n < p^{s+1}$ and $[x]$ denotes the greatest integer less than or equal to x.
5. If n is a natural number and r is a positive integer such that $0 \le r \le n$, then
   $$^nP_r = \frac{n!}{(n-r)!}$$
6. (i) (*Fundamental Principle of Multiplication*): If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one

of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

EXPLANATION If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

The above principle can be extended for any finite number of jobs as stated below:

If there are n jobs $J_1, J_2, \ldots J_n$ such that job J_i can be performed independently in m_i ways; $i = 1, 2, \ldots n$. Then the total number of ways in which all the jobs can be performed is $m_1 \times m_2 \times m_3 \times \ldots \times m_n$.

(ii) (*Fundamental Principle of Addition*) If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m+n)$ ways.

7. (i) Let r and n be positive integers such that $1 \le r \le n$. Then, the number of all permutations of n distinct items or objects taken r at a time is
   $$n(n-1)(n-2)(n-3)\ldots(n-(r-1))$$
   (ii) The number of all permutations (arrangements) of n distinct objects taken all at a time is $n!$.
   (iii) The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$, is
   $$\frac{n!}{p!\,q!}$$
   (iv) The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ... pr are alike of rth kind such that $p_1 + p_2 + \ldots + p_r = n$, is
   $$\frac{n!}{p_1!\,p_2!\,p_3! \ldots p_r!}$$
   (v) The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is
   $$\frac{n!}{p!\,q!}$$
   (vi) Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is
   $$\frac{(p_1 + p_2 + \ldots + p_r)!}{p_1!\,p_2!\,p_3! \ldots p_r!}$$

17

COMBINATIONS

17.1 INTRODUCTION

In the previous chapter, we have studied arrangements of a certain number of objects by taking some of them or all at a time. Most of the times we are not interested in arranging the objects, but we are more concerned in selecting a number of objects from given objects. In other words, we do not want to specify the ordering of selected objects. For example, a company may want to select 3 persons out of 10 applicants, a student may want to choose three books from his library at a time etc.

Suppose we want to select three persons out of 4 persons A, B, C and D. We may choose A, B, C or A, B, D or A, C, D or B, C, D. Note that we have not listed A, B, C; B, C, A; C, A, B; B, A, C; C, B, A and A, C, B separately here, because they represent the same selection A, B, C. But they give rise to different arrangements. It is evident from the above discussion that in a selection the order in which objects are arranged is immaterial.

17.2 COMBINATIONS

COMBINATIONS *Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.*

ILLUSTRATION 1 *List the different combinations formed of three letters A, B, C taken two at a time.*

SOLUTION The different combinations formed of three letters A, B, C are :

$$AB, \quad AC, \quad BC.$$

ILLUSTRATION 2 *Write all combinations of four letters A, B, C, D taken two at a time.*

SOLUTION Various combinations of two letters out of four letters A, B, C, D are:

$$AB, \quad AC, \quad AD, \quad BC, \quad BD, \quad CD.$$

DIFFERENCE BETWEEN A PERMUTATION AND COMBINATION

I In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

II In a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential. For example, A, B and B, A are same as combinations but different as permutations.

III Practically to find the permutations of n different items, taken r at a time, we first select r items from n items and then arrange them. So, usually the number of permutations exceeds the number of combinations.

IV Each combination corresponds to many permutations. For example, the six permutations ABC, ACB, BCA, BAC, CBA and CAB correspond to the same combination ABC.

REMARK *Generally we use the word 'arrangements' for permutations and the word 'selections' for combinations.*

NOTATION *The number of all combinations of n objects, taken r at a time is generally denoted by C(n, r) or,* ${}^nC_r$ *or,* $\binom{n}{r}$.

Thus, ${}^nC_r$ or $C(n, r)$ = Number of ways of selecting r objects from n objects.

Clearly, ${}^nC_r$ is defined only when n and r are non-negative integers such that $0 \le r \le n$.

THEOREM *The number of all combinations of n distinct objects, taken r at a time is given by*

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

PROOF Let the number of combinations of n distinct objects taken r at a time be x. Consider one of these x ways. There are r objects in this selection which can be arranged in $r!$ ways. Thus, each of the x combinations gives rise to $r!$ permutations. So, x combinations will give rise to $x \times (r!)$ permutations. Consequently, the number of permutations of n things, taken r at a time is $x \times (r!)$. But this number is also equal to ${}^nP_r$.

$$\therefore \quad x\,(r!) = {}^nP_r \Rightarrow x = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} \qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow \quad {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Q.E.D.

REMARK 1 *We have,*

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\Rightarrow \quad {}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots 3.2.1}{\{(n-r)(n-r-1)\dots 3.2.1\}\{1.2.3.\dots r\}}$$

$$\Rightarrow \quad {}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3.\dots r}$$

Sometimes this form of ${}^nC_r$ *is also very convenient to use.*

REMARK 2 *We have,*

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Putting $r = n$, *we obtain* ${}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{n!\,0!} = 1 \; [\because 0! = 1]$

Putting $r = 0$, *we obtain* ${}^nC_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1$

Thus, ${}^nC_n = {}^nC_0 = 1$.

REMARK 3 *We have,*

$${}^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{1}{r!}\left(\frac{n!}{(n-r)!}\right) = \frac{{}^nP_r}{r!}.$$

17.3 PROPERTIES OF ${}^nC_r$ OR, $C(n, r)$

In this section, we shall discuss some important properties of ${}^nC_r$.

PROPERTY 1 For $0 \le r \le n$, we have ${}^nC_r = {}^nC_{n-r}$

PROOF We have,

$${}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

REMARK 1 *The use of this property simplifies the calculation of* ${}^nC_r$ *when r is large.*

For example, if we want to calculate ${}^{20}C_{19}$, *by using this property. We have*

$${}^{20}C_{19} = {}^{20}C_{20-19} = {}^{20}C_1 = 20.$$

REMARK 2 *The above property can be restated as follows :*

If x and y are non-negative integers such that $x + y = n$, *then* ${}^nC_x = {}^nC_y$

This can also be stated as : ${}^nC_x = {}^nC_y \Rightarrow x = y$, *or* $x + y = n$

ILLUSTRATION 1 *If* ${}^nC_7 = {}^nC_4$, *find the value of n.*

SOLUTION We know that: ${}^nC_x = {}^nC_y \Leftrightarrow x + y = n$ or $x = y$.

$$\therefore \quad {}^nC_7 = {}^nC_4 \Rightarrow n = 7 + 4 = 11$$

PROPERTY 2 *Let n and r be non-negative integers such that* $r \le n$. *Then,*

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$$

PROOF We have,

$$\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-(r-1))!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\,r\,(r-1)!} + \frac{n!}{(n-r+1)(n-r)!\,(r-1)!} \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\} \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{n-r+1+r}{r(n-r+1)}\right\} \\
&= \frac{n!\,(n+1)}{(n-r)!\,(r-1)!\,r\,(n-r+1)} \\
&= \frac{(n+1)\,n!}{(n-r+1)(n-r)!\,r\,(r-1)!} \\
&= \frac{(n+1)!}{(n-r+1)!\,r!} = \frac{(n+1)!}{((n+1)-r)!\,r!} = {}^{n+1}C_r
\end{aligned}$$

REMARK 3 *This property is known as Pascal's rule and it can also be proved by giving combinatorial arguments.*

ILLUSTRATION 2 *Find the value of the expression* ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$

SOLUTION We have,

$${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$$

$$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$$

$$= {}^{47}C_3 + {}^{47}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$$

$= ({}^{47}C_3 + {}^{47}C_4) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$ $\quad [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]$

$= ({}^{48}C_3 + {}^{48}C_4) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$

$= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3$ $\quad [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]$

$= ({}^{49}C_3 + {}^{49}C_4) + {}^{50}C_3 + {}^{51}C_3$

$= {}^{50}C_4 + {}^{50}C_3 + {}^{51}C_3$ $\quad [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]$

$= ({}^{50}C_3 + {}^{50}C_4) + {}^{51}C_3$

$= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4$ $\quad [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]$ $\quad [\because {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]$

PROPERTY 3 *Let n and r be non-negative integers such that $1 \le r \le n$. Then,*

$${}^{n}C_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$$

PROOF We have,

$${}^{n}C_r = \frac{n!}{(n-r)!\, r!}$$

$$\Rightarrow \quad {}^{n}C_r = n \frac{(n-1)!}{\{(n-1)-(r-1)\}!\, r\, (r-1)!}$$

$$\Rightarrow \quad {}^{n}C_r = \frac{n}{r} \frac{(n-1)!}{\{(n-1)-(r-1)\}!\, (r-1)!} = \frac{n}{r}\, {}^{n-1}C_{r-1}$$

REMARK 4 *This property is very useful to find the value of ${}^{n}C_r$.*

For example, ${}^{10}C_3 = \frac{10}{3} \times {}^{9}C_2 = \frac{10}{3} \times \frac{9}{2} \times {}^{8}C_1 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1}\, {}^{7}C_0 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times 1 = 120$

REMARK 5 *By using the above property, we obtain that*

$${}^{n}C_r = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \frac{n-2}{r-2} \cdots \frac{n-(r-1)}{2} \cdot \frac{n-(r-1)}{1}$$

For example, ${}^{9}C_4 = \frac{9}{4} \times \frac{8}{3} \times \frac{7}{2} \times \frac{6}{1} = 126.$

PROPERTY 4 *If $1 \le r \le n$, then*

$$n \cdot {}^{n-1}C_{r-1} = (n-r+1)\, {}^{n}C_{r-1}$$

PROOF We have,

$$n \cdot {}^{n-1}C_{r-1} = n \cdot \frac{(n-1)!}{\{(n-1)-(r-1)\}!\, (r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = \frac{n!}{(n-r)!\, (r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = \frac{(n-r+1) \cdot n!}{(n-r+1)\, (n-r)!\, (r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1) \left[\frac{n!}{(n-r+1)!\, (r-1)!}\right]$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1) \left[\frac{n!}{\{n-(r-1)\}!\, (r-1)!}\right]$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1)\, {}^{n}C_{r-1}$$

PROPERTY 5 ${}^{n}C_x = {}^{n}C_y \Rightarrow x = y$ *or,* $x + y = n$.

PROOF We have,

$${}^{n}C_x = {}^{n}C_y$$

$$\Rightarrow \quad {}^{n}C_x = {}^{n}C_y = {}^{n}C_{n-y} \qquad [\because {}^{n}C_y = {}^{n}C_{n-y}]$$

$$\Rightarrow \quad x = y \text{ or } x = n - y$$

$$\Rightarrow \quad x = y \text{ or } x + y = n.$$

REMARK 6 *If ${}^{n}C_x = {}^{n}C_y$ and $x \ne y$, then $x + y = n$.*

ILLUSTRATION 3 *If ${}^{n}C_{15} = {}^{n}C_8$, find the value of ${}^{n}C_{21}$.*

SOLUTION We have,

$${}^{n}C_{15} = {}^{n}C_8 \Rightarrow n = (15+8) = 23 \qquad [{}^{n}C_x = {}^{n}C_y \Rightarrow x + y = n]$$

$${}^{n}C_{21} = {}^{23}C_{21} = {}^{23}C_{23-21} \qquad [\because {}^{n}C_r = {}^{n}C_{n-r}]$$

$$= {}^{23}C_2 = \frac{23}{2} \times \frac{22}{1} \times {}^{21}C_0 \qquad \left[\because {}^{n}C_r = \frac{n}{r} \times \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}\right]$$

$$= \frac{23}{2} \times \frac{22}{1} \times 1 \qquad [\because {}^{n}C_0 = 1]$$

$$= 23 \times 11 = 253$$

ILLUSTRATION 4 *If ${}^{10}C_x = {}^{10}C_{x+4}$, find the value of x.*

SOLUTION We have, ${}^{10}C_x = {}^{10}C_{x+4} \Rightarrow x + x + 4 = 10 \Rightarrow 2x = 6 \Rightarrow x = 3.$

PROPERTY 6 *If n is an even natural number, then the greatest of the values*

$${}^{n}C_0, {}^{n}C_1, {}^{n}C_2, \ldots, {}^{n}C_n \text{ is } {}^{n}C_{n/2}$$

If n is an odd natural number, then the greatest of the values

$${}^{n}C_0, {}^{n}C_1, {}^{n}C_2, \ldots {}^{n}C_n \text{ is } {}^{n}C_{n-1/2} = {}^{n}C_{n+1/2}$$

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Evaluate the following:* (i) ${}^{10}C_8$ (ii) ${}^{100}C_{98}$ (iii) ${}^{52}C_{52}$

SOLUTION We have,

(i) $\quad {}^{10}C_8 = {}^{10}C_{10-8} \qquad [\because {}^{n}C_r = {}^{n}C_{n-r}]$

$$\Rightarrow \quad {}^{10}C_8 = {}^{10}C_2 = \frac{10}{2} \times \frac{9}{1} \times {}^{8}C_0 \qquad \left[\because {}^{n}C_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow \quad {}^{10}C_8 = \frac{10}{2} \times \frac{9}{1} \times 1 \qquad [\because {}^{n}C_0 = 1]$$

$$\Rightarrow \quad {}^{10}C_8 = 45 \qquad [\because {}^{n}C_r = {}^{n}C_{n-r}]$$

(ii) $\quad {}^{100}C_{98} = {}^{100}C_{100-98} \qquad [\because {}^{n}C_r = {}^{n}C_{n-r}]$

$$\Rightarrow \quad {}^{100}C_{98} = {}^{100}C_2 = \frac{100}{2} \times \frac{99}{1} \times {}^{98}C_0 \qquad \left[\because {}^{n}C_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$$

$$\Rightarrow \quad {}^{100}C_{98} = \frac{100}{2} \times \frac{99}{1} \times 1 \qquad [\because {}^{n}C_0 = 1]$$

$$\Rightarrow \quad {}^{100}C_{98} = 4950$$

(iii) $\quad {}^{52}C_{52} = 1 \qquad [\because {}^{n}C_n = 1]$

EXAMPLE 2 *If* ${}^{n}C_8 = {}^{n}C_6$, *find* ${}^{n}C_2$.

SOLUTION If ${}^{n}C_x = {}^{n}C_y$ and $x \neq y$, then $x + y = n$.

$\therefore \quad {}^{n}C_8 = {}^{n}C_6 \Rightarrow n = (8+6) = 14$

Now $\quad {}^{n}C_2 = {}^{14}C_2 = \frac{14}{2} \times \frac{13}{1} \times {}^{12}C_0 \qquad \left[\because {}^{n}C_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}\right]$

$= \frac{14}{2} \times \frac{13}{1} \times 1 \qquad [\because {}^{n}C_0 = 1]$

$= 91$

EXAMPLE 3 *If* ${}^{n}P_r = 720$ *and* ${}^{n}C_r = 120$, *find* r.

SOLUTION We know that

$${}^{n}C_r = \frac{{}^{n}P_r}{r!}$$

$$\Rightarrow \quad 120 = \frac{720}{r!}$$

$$\Rightarrow \quad r! = 6 \Rightarrow r! = 3! \Rightarrow r = 3. \qquad [\text{On putting the value of } {}^{n}P_r \text{ and } {}^{n}C_r]$$

EXAMPLE 4 *If the ratio* ${}^{2n}C_3 : {}^{n}C_3$ *is equal to* $11 : 1$, *find* n.

SOLUTION We have,

$${}^{2n}C_3 : {}^{n}C_3 = 11 : 1$$

$$\Rightarrow \quad \frac{{}^{2n}C_3}{{}^{n}C_3} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{\frac{(2n)!}{(2n-3)!\,3!}}{\frac{n!}{(n-3)!\,(3!)}} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)(2n-1)(2n-2)(2n-3)!}{(2n-3)!} \times \frac{(n-3)!}{n(n-1)(n-2)(n-3)!} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{(2n)(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1}$$

$$\Rightarrow \quad \frac{4(2n-1)}{n-2} = \frac{11}{1} \Rightarrow 8n - 4 = 11n - 22 \Rightarrow 3n = 18 \Rightarrow n = 6$$

EXAMPLE 5 *Prove that the product of* r *consecutive positive integers is divisible by* $r!$.

SOLUTION Let the r consecutive positive integers be $(n+1), (n+2), (n+3), \ldots, (n+r)$.

Then,

$$\text{Product} = (n+1)(n+2)(n+3)\ldots(n+r)$$

$$= \frac{n!\,(n+1)(n+2)(n+3)\ldots(n+r)}{n!}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)(n+2)\ldots(n+r)}{n!}$$

$$= \frac{(n+r)!}{n!} = \frac{(n+r)!}{r!\,\{(n+r)-r\}!}\,(r!)$$

$$= \left({}^{n+r}C_r\right) r!, \text{ which is divisible by } r! \qquad [\because {}^{n+r}C_r \text{ is an integer}]$$

EXAMPLE 6 *Prove that* : ${}^{2n}C_n = \frac{2^n\,[1 \cdot 3 \cdot 5 \ldots (2n-1)]}{n!}$.

SOLUTION We have,

$${}^{2n}C_n = \frac{2n!}{(2n-n)!\,n!} = \frac{(2n)!}{n!\,n!}$$

$$= \frac{(2n)(2n-1)(2n-2)\ldots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)][2 \cdot 4 \cdot 6 \ldots 2n]}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)] \times 2^n\,[1 \cdot 2 \cdot 3 \ldots n]}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)] \times 2^n \times n!}{n!\,n!} = 2^n\,\frac{[1 \cdot 3 \cdot 5 \ldots (2n-1)]}{n!}$$

EXAMPLE 7 *If* ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, *find* n.

SOLUTION We have,

$${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$$

$$\Rightarrow \quad \frac{{}^{n+2}C_8}{{}^{n-2}P_4} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)!}{8!\,(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)(n+1)\,n\,(n-1)(n-2)!}{8!} \times \frac{1}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad (n+2)(n+1)\,n\,(n-1) = \frac{57}{16} \times 8! = \frac{19 \times 3}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$\Rightarrow \quad (n+2)(n+1)(n-1)\,n = 143640$$

$$\Rightarrow \quad (n-1)\,n\,(n+1)(n+2) = 19 \times 3 \times 7 \times 6 \times 5 \times 4 \times 3$$

$$\Rightarrow \quad (n-1)\,n\,(n+1)(n+2) = 19 \times (3 \times 7) \times (6 \times 3) \times (4 \times 5)$$

$$\Rightarrow \quad (n-1)\,n\,(n+1)(n+2) = 18 \times 19 \times 20 \times 21$$

$$\Rightarrow \quad n - 1 = 18 \Rightarrow n = 19$$

EXAMPLE 8 *If* ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$ *and* ${}^{n}C_{r+1} = 126$, *then find* ${}^{r}C_2$.

SOLUTION We know that

$$\frac{{}^{n}C_r}{{}^{n}C_{r+1}} = \frac{r+1}{n-r}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{84}{126} = \frac{2}{3} \Rightarrow 2n - 5r = 3 \qquad \ldots(i)$$

Replacing r by $(r-1)$ in $\frac{{}^{n}C_r}{{}^{n}C_{r+1}}$, we get

$$\frac{{}^{n}C_{r-1}}{{}^{n}C_r} = \frac{r}{n-(r-1)}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{36}{84}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7}$$

$$\Rightarrow \quad 3n - 10r = -3 \qquad \text{...(iii)}$$

Solving (i) and (iii), we get $r = 3$.

$$\therefore \quad {}^rC_2 = {}^3C_2 = \frac{3!}{(3-2)!\,2!} = 3$$

NOTE *Students are advised to learn that*

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \text{ as it is a very useful result.}$$

EXAMPLE 8 *If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$, find the values of n and r.*

SOLUTION We have,

$${}^nP_r = {}^nP_{r+1}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$$

$$\Rightarrow \quad \frac{1}{(n-r)(n-r-1)!} = \frac{1}{(n-r-1)!}$$

$$\Rightarrow \quad n - r = 1 \qquad \text{...(i)}$$

and, $\quad {}^nC_r = {}^nC_{r-1}$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r\,(r-1)!} = \frac{n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{n-r+1}$$

$$\Rightarrow \quad n - r + 1 = r \Rightarrow n - 2r = -1 \qquad \text{...(ii)}$$

Solving (i) and (ii), we obtain $n = 3$ and $r = 2$.

EXERCISE 17.1

1. Evaluate the following:
 (i) ${}^{14}C_3$ (ii) ${}^{12}C_{10}$ (iii) ${}^{35}C_{35}$ (iv) ${}^{n+1}C_n$ (v) $\sum_{r=1}^{5} {}^5C_r$
2. If ${}^nC_{12} = {}^nC_5$, find the value of n.
3. If ${}^nC_4 = {}^nC_6$, find ${}^{12}C_n$.
4. If ${}^nC_{10} = {}^nC_{12}$, find ${}^{23}C_n$.
5. If ${}^{24}C_x = {}^{24}C_{2x+3}$, find x.
6. If ${}^{18}C_x = {}^{18}C_{x+2}$, find x.
7. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find r.
8. If ${}^8C_r - {}^7C_3 = {}^7C_2$, find r.
9. If ${}^{15}C_r : {}^{15}C_{r-1} = 11 : 5$, find r.
10. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, find n.

11. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, find r.
12. Prove that: ${}^{4n}C_{2n} : {}^{2n}C_n = [1 \cdot 3 \cdot 5 \ldots (4n-1)] : [1 \cdot 3 \cdot 5 \ldots (2n-1)]^2$.
13. If ${}^{2n}C_3 : {}^nC_2 = 44 : 3$, find n.
14. If ${}^{16}C_r = {}^{16}C_{r+2}$, find ${}^rC_4$.
15. Evaluate ${}^{20}C_5 + \sum_{r=2}^{5} {}^{25-r}C_4$
16. Prove that the product of $2n$ consecutive negative integers is divisible by $(2n)!$
17. For all positive integers n, show that ${}^{2n}C_n + {}^{2n}C_{n-1} = \frac{1}{2}({}^{2n+2}C_{n+1})$.
18. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then find n.
19. If $\alpha = {}^mC_2$, then find the value of ${}^\alpha C_2$.
20. Let r and n be positive integers such that $1 \le r \le n$. Then prove the following:
 (i) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ (ii) $n \cdot {}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$
 (iii) $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$ (iv) ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$

ANSWERS

1. (i) 364	(ii) 66	(iii) 1		(iv) $(n+1)$		(v) 31
2. 17	3. 66	4. 23	5. 7	6. 8	7. 3	
8. 3, 5	9. 5	10. 19	11. 7	13. 6	14. 35	
15. 42504	18.	14, 7		19. $\frac{(m+1)(m)(m-1)(m-2)}{8}$		

HINTS TO NCERT & SELECTED PROBLEMS

16. Let $(-r), (-r-1), (-r-2), \ldots, (-r-2n+1)$ be $2n$ consecutive negative integers. Then,

$$\text{Product} = (-1)^{2n}\, r(r+1)(r+2)\ldots(r+2n-1)$$

$$= \frac{(r-1)!\,(r)(r+1)\ldots(r+2n-1)}{(r-1)!}$$

$$= \frac{(r+2n-1)!}{(r-1)!} = \frac{(r+2n-1)!}{(r-1)!\,(2n)!}(2n)! = ({}^{r+2n-1}C_{2n})(2n)!$$

Clearly, it is divisible by $(2n)!$

17.4 PRACTICAL PROBLEMS ON COMBINATIONS

In this section, we intend to discuss some problems in real life where the formula for ${}^nC_r$ and its meaning can be applied.

EXAMPLE 1 *From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done?*

SOLUTION Required number of ways $= {}^{32}C_4 = \frac{32!}{28!\,4!}$

EXAMPLE 2 *Three gentlemen and three ladies are candidates for two vacancies. A voter has to vote for two candidates. In how many ways can one cast his vote?*

SOLUTION Clearly, there are 6 candidates and a voter has to vote for any two of them. So, the required number of ways is the number of ways of selecting 2 out of 6 i.e. ${}^6C_2$.

Hence, the required number of ways $= {}^6C_2 = \frac{6!}{2!\,4!} = 15$.

EXAMPLE 3 *If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?*

SOLUTION It is to note here that, when two persons shake hands, it is counted as one handshake, not two. So, this is a problem on combinations.

The total number of handshakes is same as the number of ways of selecting 2 persons among 12 persons $= {}^{12}C_2 = \frac{12!}{10! \times 2!} = 66$.

EXAMPLE 4 *A question paper has two parts, Part A and Part B, each containing 10 questions. If the student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions ?*

SOLUTION There are 10 questions in Part A out of which 8 questions can be chosen in ${}^{10}C_8$ ways. Similarly, 5 questions can be chosen from part B containing 10 questions in ${}^{10}C_5$ ways.

Hence, the total number of ways of selecting 8 questions from part A and 5 from part B

$$= {}^{10}C_8 \times {}^{10}C_5 = \frac{10!}{8!\,2!} \times \frac{10!}{5! \times 5!} = 11340.$$

EXAMPLE 5 *In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women ?*

SOLUTION Three men out of 6 men can be selected in ${}^6C_3$ ways. Two women out of 5 women can be selected in ${}^5C_2$ ways. Therefore, by the fundamental principle of counting, 3 men out of 6 men and 2 women out of 5 women can be selected in

$${}^6C_3 \times {}^5C_2 = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 200 \text{ ways.}$$

EXAMPLE 6 *In how many ways can a cricket eleven be chosen out of a batch of 15 players if*

(i) there is no restriction on the selection;

(ii) a particular player is always chosen;

(iii) a particular player is never chosen?

SOLUTION

(i) The total number of ways of selecting 11 players out of 15 is

$${}^{15}C_{11} = {}^{15}C_{15-11} = {}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365$$

(ii) If a particular player is always chosen. This means that 10 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{10} = {}^{14}C_{14-10} = {}^{14}C_4 = 1001$

(iii) If a particular player is never chosen. This means that 11 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{11} = {}^{14}C_{14-11} = {}^{14}C_3 = 364$

EXAMPLE 7 *A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee ? In how many of these committees (i) the women are in majority (ii) the men are in majority ?*

SOLUTION There are 9 women and 8 men. A committee of 12, consisting of at least 5 women, can be formed by choosing :

(I) 5 women and 7 men

(II) 6 women and 6 men

(III) 7 women and 5 men

(IV) 8 women and 4 men

(V) 9 women and 3 men

Total number of ways of forming the committee

$$= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$$

$$= 126 \times 8 + 84 \times 28 + 36 \times 56 + 9 \times 70 + 1 \times 56 = 6062$$

Clearly, women are in majority in (III), (IV) and (V) cases as discussed above.

So, total number of committees in which women are in majority

$$= {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3 = 36 \times 56 + 9 \times 70 + 1 \times 56 = 2702$$

Clearly, men are in majority in only (I) case as discussed above.

So, total number of committees in which men are in majority

$$= {}^9C_5 \times {}^8C_7 = 126 \times 8 = 1008.$$

EXAMPLE 8 *A committee of three persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?* **[NCERT]**

SOLUTION There are 5 persons (2 men and 3 women). In order consitute a committee of 3 persons we need to select three persons out of given 5 persons. This can be done in ${}^5C_3$ ways.

So, the committee can be formed in ${}^5C_3 = \frac{5!}{3!\,2!} = 10$ ways.

Now, 1 man can be selected from 2 men in ${}^2C_1$ ways and 2 women can be selected from 3 women in ${}^3C_2$ ways.

Therefore, required number of committees is ${}^2C_1 \times {}^3C_2 = 2 \times 3 = 6$

EXAMPLE 9 *What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these*

(i) four cards are of the same suit,

(ii) four cards belong to four different suits,

(iii) four cards are face cards,

(iv) two are red cards and two are black cards,

(v) cards are of the same colour? **[NCERT]**

SOLUTION Four cards can be chosen from 52 playing cards in ${}^{52}C_4$ ways i.e.,

$$\frac{52!}{48!\,4!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4} = 270725 \text{ ways.}$$

(i) There are four suits (diamond, spade, club and heart) of 13 cards each. Therefore, there are ${}^{13}C_4$ ways of choosing 4 diamond cards, ${}^{13}C_4$ ways of choosing 4 club cards, ${}^{13}C_4$ ways of choosing 4 spade cards and ${}^{13}C_4$ ways of choosing heart cards.

$\therefore$ Required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$

$$= 4 \times {}^{13}C_4$$

$$= 4 \times \frac{13!}{9!\,4!} = 2860$$

(ii) There are 13 cards in each suit. Four cards drawn belong to four different suits means one card is drawn from each suit. Out of 13 diamond cards one card can be drawn in $^{13}C_1$ ways. Similarly, there are $^{13}C_1$ ways of choosing one club card, $^{13}C_1$ ways of choosing one spade card and $^{13}C_1$ ways of choosing one heart card.

∴ Number of ways of selecting one card from each suit

$$= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$$

(iii) There are 12 face cards out of which 4 cards can be chosen in $^{12}C_4$ ways.

∴ Required number of ways $= {}^{12}C_4 = \frac{12!}{4!\,8!} = 495$

(iv) There are 26 red cards and 26 black cards. Therefore, 2 red cards can be chosen in $^{26}C_2$ ways and 2 black cards can be chosen in $^{26}C_2$ ways. Hence, 2 red and 2 black cards can be chosen in $^{26}C_2 \times {}^{26}C_2$ ways i.e., $\left(\frac{26!}{24!\,2!}\right)^2 = (325)^2 = 105625$ ways.

(v) Out of 26 red cards, 4 red cards can be chosen in $^{26}C_4$ ways. Similarly, 4 black cards can be chosen in $26C_4$ ways.

Hence, 4 red or 4 black cards can be chosen in $^{26}C_4 + {}^{26}C_4$ ways i.e.,

$$2 \times {}^{26}C_4 = 2 \times \frac{26!}{4!\,22!} = 29900 \text{ ways.}$$

EXAMPLE 10 *A person wishes to make up as many different parties as he can out of his 20 friends such that each party consists of the same number of persons. How many friends should he invite?*

SOLUTION Suppose he invites r friends at a time. Then the total number of parties is $^{20}C_r$. We have to find the maximum value of $^{20}C_r$ which is for $r = 10$ (if n is even, then ^nC_r is maximum for $r = n/2$). Hence, he should invite 10 friends at a time in order to form the maximum number of parties.

EXAMPLE 11 *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included?*

SOLUTION The committee can be formed in the following ways:

(i) By selecting 2 men and 1 woman

(ii) By selecting 1 man and 2 women

Now, 2 men out of 5 men and 1 woman out of 2 woman can be chosen in $^5C_2 \times {}^2C_1$ ways.

And, 1 man out of 5 men and 2 women out of 2 women can be chosen in $^5C_1 \times {}^2C_2$ ways.

∴ Total number of ways of forming the committee

$$= {}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 = 20 + 5 = 25.$$

EXAMPLE 12 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers? Assume that the team of 11 players requires 5 batsmen, 3 all-rounder, 2 bowlers and 1 wicket keeper.*

SOLUTION The selection of team is divided into four phases:

I Selection of 5 batsmen out of 10. This can be done in $^{10}C_5$ ways.

II Selection of 3 all-rounders out of 5. This can be done in $^5C_3$ ways.

III Selection of 2 bowlers out of 8. This can be done in $^8C_2$ ways.

IV Selection of one wicket keeper out of 2. This can be done in $^2C_1$ ways.

The selection of team is completed by completing all the four phases.

∴ The team can be selected in $^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1 = 141120$ ways.

EXAMPLE 13 *A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done, when*

(i) at least two ladies are included; (ii) at most two ladies are included?

SOLUTION

(i) A committee of 5 persons, consisting of at least two ladies, can be formed in the following ways:

I Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in $^4C_2 \times {}^6C_3$ ways.

II Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in $^4C_3 \times {}^6C_2$ ways.

III Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in $^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case, therefore, by the fundamental principle of addition, the total number of ways of forming the committee $= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1 = 120 + 60 + 6 = 186$

(ii) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways:

I Selecting 5 gents only out of 6. This can be done in $^6C_5$ ways.

II Selecting 4 gents only out of 6 and one lady out of 4. This can be done in $^6C_4 \times {}^4C_1$ ways.

III Selecting 3 gents only out of 6 and two ladies out of 4. This can be done is $^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case, so, the total number of ways of forming the committee $= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186.$

EXAMPLE 14 *A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?*

SOLUTION The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways:

I By selecting 2 red balls out of 5 and 4 white balls out of 6. This can be done in $^5C_2 \times {}^6C_4$ ways.

II By selecting 3 red balls out of 5 and 3 white balls out of 6. This can be done in $^5C_3 \times {}^6C_3$ ways.

III By selecting 4 red balls out of 5 and 2 white balls out of 6. This can be done in $^5C_4 \times {}^6C_2$ ways.

Since the selection of 6 balls can be completed in any one of the above ways. Hence, by the fundamental principle of addition, the total number of ways to select the balls

$$= {}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2 = 10 \times 15 + 10 \times 20 + 5 \times 15 = 425.$$

EXAMPLE 15 *For the post of 5 teachers, there are 23 applicants, 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made ?*

SOLUTION Clearly, there are 7 SC candidates and 16 other candidates. We have to select 2 out of 7 SC candidates and 3 out of 16 other candidates. This can be done in ${}^7C_2 \times {}^{16}C_3$ ways.

$\therefore$ The number of ways of making the selection $= {}^7C_2 \times {}^{16}C_3 = 11760$.

EXAMPLE 16 *How many triangles can be formed by joining the vertices of a hexagon ?*

SOLUTION There are 6 vertices of a hexagon. One triangle is formed by selecting a group of 3 vertices from given 6 vertices. This can be done in ${}^6C_3$ ways.

$\therefore$ Number of triangles $= {}^6C_3 = \dfrac{6!}{3!\,3!} = 20$.

EXAMPLE 17 *How many diagonals are there in a polygon with n sides ?*

SOLUTION *A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of a n sided polygon taken two at a time*

= Number of ways of selecting 2 out of n

$= {}^nC_2 = \dfrac{n(n-1)}{2}$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon $= \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$.

EXAMPLE 18 *How many chords can be drawn through 21 points on a circle?*

SOLUTION A chord is obtained by joining any two points on a circle. Therefore, total number of chords drawn through 21 points is same as the number of ways of selecting 2 points out of 21 points. This can be done in ${}^{21}C_2$ ways.

Hence, total number of chords $= {}^{21}C_2 = \dfrac{21!}{19!\,2!} = 21 \times 10 = 210$

EXAMPLE 19 *A polygon has 44 diagonals. Find the number of its sides.*

SOLUTION Let there be n sides of the polygon. We know that the number of diagonals of n sided polygon is $\dfrac{n(n-3)}{2}$

$\therefore \dfrac{n(n-3)}{2} = 44 \Rightarrow n^2 - 3n - 88 = 0 \Rightarrow (n-11)(n+8) = 0 \Rightarrow n = 11$ $(\because n > 0)$

Hence, there are 11 sides of the polygon.

EXAMPLE 20 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelograms formed.*

SOLUTION A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Two straight lines from the set of m parallel lines can be chosen in ${}^mC_2$ ways and two straight lines from the set of n parallel lines can be chosen in ${}^nC_2$ ways. Hence, the number of parallelograms formed

$= {}^mC_2 \times {}^nC_2 = \dfrac{m(m-1)}{2} \times \dfrac{n(n-1)}{2} = \dfrac{mn(m-1)(n-1)}{4}$

EXAMPLE 21 *There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the (i) number of straight lines obtained from the pairs of these points; (ii) number of triangles that can be formed with the vertices as these points.*

SOLUTION

(i) Number of straight lines formed joining the 10 points, taking 2 at a time

$= {}^{10}C_2 = \dfrac{10!}{2!\,8!} = 45$.

Number of straight lines formed by joining the four points, taking 2 at a time

$= {}^4C_2 = \dfrac{4!}{2!\,2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.

$\therefore$ Required number of straight lines $= 45 - 6 + 1 = 40$.

(ii) Number of triangles formed by joining the points, taking 3 at a time

$= {}^{10}C_3 = \dfrac{10!}{3!\,7!} = 120$.

Number of triangles formed by joining the 4 points, taken 3 at a time

$= {}^4C_3 = {}^4C_1 = 4$.

But, 4 collinear points cannot form a triangle when taken 3 at a time. So,
Required number of triangles $= 120 - 4 = 116$.

EXAMPLE 22 *In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

SOLUTION The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting ${}^{11}C_2$ points, we get only one point B. Hence, the number of intersection points of the lines is ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$.

EXAMPLE 23 *From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen ?* **[NCERT]**

SOLUTION We have the following possibilities:

(I) Three particular students join the excursion party.
In this case, we have to choose 7 students from the remaining 22 students. This can be done in ${}^{22}C_7$ ways.

(II) Three particular students do not join the excursion party.
In this case, we have to choose 10 students from the remaining 22 students. This can be done in ${}^{22}C_{10}$ ways.

Hence, the required number of ways $= {}^{22}C_7 + {}^{22}C_{10} = 817190$.

EXAMPLE 24 *A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry Part II, unless Chemistry Part I is also borrowed. In how many ways can he choose the three books to be borrowed ?*

SOLUTION We have the following two possibilities :

(i) When Chemistry part I is borrowed.
In this case the boy may borrow Chemistry Part II. So, he has to select now two books out of the remaining 7 books of his interest. This can be done in $^7C_2$ ways.

(ii) When Chemistry part I is not borrowed :
In this case the boy does not want to borrow Chemistry Part II. So, he has to select three books from the remaining 6 books. This can be done in $^6C_3$ ways.

Hence, the required number of ways $= {^7C_2} + {^6C_3} = 21 + 20 = 41$.

EXAMPLE 25 *In how many ways can 7 plus (+) signs and 5 minus (−) signs be arranged in a row so that no two minus signs are together ?*

SOLUTION The plus signs can be arranged in only one way, because all are identical, as shown below:

	+		+		+		+		+		+		+	

A blank box in the above arrangement shows available space for the minus signs. Since there are 7 plus signs, the number of blank boxes is therefore 8. The five minus signs are now to be arranged in the 8 boxes so that no two of them are together. Now, 5 boxes out of 8 can be chosen in $^8C_5$ ways. Since all minus signs are identical, so 5 minus signs can be arranged in 5 chosen boxes in only one way. Hence, the number of possible arrangements $= 1 \times {^8C_5} \times 1 = 56$.

EXAMPLE 26 *In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together ?*

SOLUTION In order that no two books on Hindi are together, we must first arrange all books in English in a row. Since all English books are identical, so they can be arranged in a row in only one way as shown below:

$$\times E \times E \times E \times E \times \ldots \times E \times E$$

Here E denotes the position of an English book and $\times$ that of a Hindi book.

Since there are 21 books on English, the number places mark $\times$ are therefore 22. Now, 19 books on Hindi are to be arranged in these 22 places so that no two of them are together. Out of 22 places 19 places for Hindi books can be chosen in $^{22}C_{19}$ ways. Since all books on Hindi are identical, so 19 books on Hindi can be arranged in 19 chosen places in only one way. Hence, the required number of ways $= 1 \times {^{22}C_{19}} \times 1 = 1540$.

EXERCISE 17.2

1. From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways can this be done ?
2. How many different boat parties of 8, consisting of 5 boys and 3 girls, can be made from 25 boys and 10 girls ?
3. In how many ways can a student choose 5 courses out of 9 courses if 2 courses are compulsory for every student ?
4. In how many ways can a football team of 11 players be selected from 16 players ? How many of these will
   (i) include 2 particular players ?
   (ii) exclude 2 particular players ?

5. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees:
   (i) a particular professor is included.
   (ii) a particular student is included.
   (iii) a particular student is excluded.
6. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 (without repetition) ?
7. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition; at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made ?
8. How many different selections of 4 books can be made from 10 different books, if
   (i) there is no restriction;
   (ii) two particular books are always selected;
   (iii) two particular books are never selected ?
9. From 4 officers and 8 jawans in how many ways can 6 be chosen (i) to include exactly one officer (ii) to include at least one officer ?
10. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the teams be constituted ?
11. A student has to answer 10 questions, choosing at least 4 from each of part A and part B. If there are 6 questions in part A and 7 in part B, in how many ways can the student choose 10 questions ?
12. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.
13. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can he choose the 7 questions?
14. There are 10 points in a plane of which 4 are collinear. How many different straight lines can be drawn by joining these points.
15. Find the number of diagonals of (i) a hexagon (ii) a polygon of 16 sides.
16. How many triangles can be obtained by joining 12 points, five of which are collinear ?
17. In how many ways can a committee of 5 persons be formed out of 6 men and 4 women when at least one woman has to be necessarily selected ?
18. In a village, there are 87 families of which 52 families have at most 2 children. In a rural development programme, 20 families are to be helped chosen for assistance, of which at least 18 families must have at most 2 children. In how many ways can the choice be made ?
19. A parallelogram is cut by two sets of m lines parallel to its sides. Find the number of parallelograms thus formed.
20. Find the number of (i) diagonals (ii) triangles formed in a decagon.
21. Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many (i) straight lines (ii) triangles can be formed by joining them ?

22. Determine the number of 5 cards combinations out of a deck of 52 cards if at least one of the 5 cards has to be a king ? [NCERT]
23. We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can the selection be made ?
24. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls? [NCERT]
25. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour. [NCERT]
26. Determine the number of 5 cards combinations out of a deck of 52 cards if there is exactly one ace in each combination. [NCERT]
27. In how many ways can one select a circket team of eleven from 17 players in which only 5 persons can bowl if each cricket team of 11 must include exactly 4 bowlers?
28. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected. [NCERT]
29. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student? [NCERT]
30. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consits of :
(i) exactly 3 girls? (ii) at least 3 girls? (iii) at most 3 girls? [NCERT]
31. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions? [NCERT]
32. A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has (i) no girl? (ii) at least one boy and one girl? (iii) at least 3 girls? [NCERT]
33. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women? [NCERT]

ANSWERS

1. 1365	2. 6375600	3. 35	4. 4368	(i) 2002	(ii) 364
5. 51300	(i) 10260	(ii) 7695	(iii) 43605	6. 11	7. 104874
8. (i) 210	(ii) 28	(iii) 70	9. (i) 224	(ii) 896	
10. $2\left({}^{20}C_5\times{}^{20}C_6\right)$		11. 266	12. 3	13. 780	14. 40
15. (i) 9	(ii) 104	16. 210	17. 246		
18. ${}^{52}C_{18}\times{}^{35}C_2+{}^{52}C_{19}\times{}^{35}C_1+{}^{52}C_{20}\times{}^{35}C_0$			19. $\left({}^{m+2}C_2\right)^2$	20. (i) 35	(ii) 120
21. (i) 144	(ii) 806	22. 886656		23. 22	24. 40
25. 2000	26. 778320	27. 3960	28. 200	29. 35	
30. (i) 504	(ii) 588	(iii) 1632	31. 420		
32. (i) 21	(ii) 441	(iii) 91	33. 10, 6		

HINTS TO NCERT & SELECTED PROBLEMS

2. Required no. of boat parties $= {}^{25}C_5\times{}^{10}C_3$.
3. Since 2 courses are compulsory, so, the student is to choose 3 courses out of the remaining 7 courses. This can be done in ${}^7C_3$ ways.

4. We have to select 11 players out of 16. So, required no. of ways $= {}^{16}C_{11}$.
(i) Since 2 particular players are always included, so, we have to select 9 players out of the remaining 14 players. This can be done in ${}^{14}C_9$ ways.
(ii) Since 2 particular players are excluded from every selection, so, we have to select 11 players from the remaining 14 players. This can be done in ${}^{14}C_{11}$ ways.
6. Total number of products = No. of ways of selecting 2 or 3 or all out of 4 numbers 3, 5, 7, 11
$$= {}^4C_2+{}^4C_3+{}^4C_4=6+4+1=11.$$
7. Since two girls who won the prizes last year are to be included in every selection. So, we have to select 8 students out of 12 boys and 8 girls, choosing at least 4 boys and at least two girls. This can be done in ${}^{12}C_6\times{}^8C_2+{}^{12}C_5\times{}^8C_3+{}^{12}C_4\times{}^8C_4$ = 104874 ways.
9. (i) Required number of ways $= {}^4C_1\times{}^8C_5$
(ii) Required no. of ways = Total no. of ways – No. of ways of selecting no officer
$$= {}^{12}C_6-{}^8C_6$$
10. Required no. of ways $= {}^{20}C_5\times{}^{20}C_6+{}^{20}C_6\times{}^{20}C_5$.
11. The various possibilities are : (i) 4 from part A and 6 from part B (ii) 5 from part A and 5 from part B (iii) 6 from part A and 4 from part B.
So, the required no. of ways $= {}^6C_4\times{}^7C_6+{}^6C_5\times{}^7C_5+{}^6C_6\times{}^7C_4=266$.
12. Required number of ways $= {}^3C_2$.
13. Required no. of ways $= {}^6C_5\times{}^6C_2+{}^6C_4\times{}^6C_3+{}^6C_3\times{}^6C_4+{}^6C_2\times{}^6C_5=780$.
14. Number of straight lines $= {}^{10}C_2-{}^4C_2+1$.
16. Number of triangles $= {}^{12}C_3-{}^5C_3$.
18. 52 families have at most 2 children, while 35 families have more than 2 children. The selection of 20 families of which at least 18 families must have at most 2 children can be made as under:
(I) 18 families out of 52 and 2 families out of 35
or, (II) 19 families out of 52 and 1 family out of 35
or, (III) 20 families out of 52.
19. Each set of parallel lines consists of (m + 2) lines and each parallelogram is formed by choosing two lines from the first set and two straight lines from the second set. Hence, the total number of parallelograms $= {}^{m+2}C_2\times{}^{m+2}C_2$.
22. Required number of combinations = Total number of 5 card combinations – Number of 5 car combinations having no king.
$$= {}^{52}C_5-{}^{48}C_5 = 886656.$$
24. Number of ways of selecting team $= {}^5C_3\times{}^4C_3=40$.
25. Number of ways of selecting 9 balls $= {}^6C_3\times{}^5C_3\times{}^5C_3=2000$.
26. Out of 4 aces one ace can be selected in ${}^4C_1$ ways and from the remaining 48 cards, four cards can be selected in ${}^{48}C_4$ ways. So, number of 5 cards combinations consisting of exactly one ace $= {}^4C_1\times{}^{48}C_4=778320$.

27. Required number of ways $= {}^5C_4 \times {}^{12}C_7$.

28. Out of 5 black and 6 red balls, 2 black and 3 red balls can be chosen in ${}^5C_2 \times {}^6C_3 = 200$ ways.

29. Required number of ways = Number of ways of selecting 3 courses out of 7 courses

$$= {}^7C_3 \text{ ways} = 35.$$

30. (i) A committee consisting of 3 girls and 4 boys can be formed in ${}^9C_4 \times {}^4C_3 = 504$ ways.

(ii) A committee consisting of at least 3 girls can be formed in

$${}^9C_4 \times {}^4C_3 + {}^9C_3 \times {}^4C_4 = 588 \text{ ways.}$$

(iii) A committee of at most 3 girls can be formed in

$${}^9C_7 \times {}^4C_0 + {}^9C_6 \times {}^4C_1 + {}^9C_5 \times {}^4C_2 + {}^9C_4 \times {}^4C_3 = 1632 \text{ ways.}$$

31. At least 3 questions can be selected in the following ways:

Part I	Part II
3	5
4	4
5	3

So, required number of ways $= {}^5C_3 \times {}^7C_5 + {}^5C_4 \times {}^7C_4 + {}^5C_5 \times {}^7C_3 = 420$.

32. (i) From a group of 4 girls and 7 boys, a team of 5 consisting of no girls can be chosen in ${}^7C_5 = 21$ ways.

(ii) A team of 5 consisting of at least one boy and one girl can be chosen in

$${}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1 = 441 \text{ ways}$$

(iii) A team of 5 consisting of at least 3 girls can be chosen in

$${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 91 \text{ ways.}$$

33. A committee of 3 persons out of 2 men and 3 women can be constituted in ${}^5C_3 = 10$ ways.

A committee of 1 man and 2 women can be constituted in ${}^2C_1 \times {}^3C_2 = 6$ ways.

17.5 MIXED PROBLEMS ON PERMUTATIONS AND COMBINATIONS

In this section, we intend to discuss some practical problems where both permutations and combinations are used as is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed ?*

SOLUTION Three consonants out of 7 and 2 vowels out of 4 can be chosen in ${}^7C_3 \times {}^4C_2$ ways. Thus, there are ${}^7C_3 \times {}^4C_2$ groups each containing 3 consonants and 2 vowels. Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

Hence, the required number of words $= ({}^7C_3 \times {}^4C_2) \times 5! = 25200$.

EXAMPLE 2 *How many four-letter words can be formed using the letters of the word 'FAILURE', so that*

(i) F is included in each word ? *(ii) F is not included in any word ?*

SOLUTION (i) There are 7 letters in the word 'FAILURE'.

To include F in every 4 letter word, we first select four letters from the 7 letters of the word 'FAILURE' such that F is included in every selection. This can be done by selecting three letters from the remaining 6 letters i.e. A, I, L, U, R, E in ${}^6C_3$ ways. Now, there are 4 letters in each of ${}^6C_3$ selections.

Consider one of these ${}^6C_3$ selections. This selection contains 4 letters which can be arranged in 4! ways.

Thus, each of ${}^6C_3$ selections provides 4! words. Hence, the total number of words $= {}^6C_3 \times 4! = 480$.

(ii) If F is not to be included in any word, then we first select 4 letters from the remaining 6 letters. This can be done in ${}^6C_4$ ways. Now, every selection has 4 letters which can be arranged in a row in 4! ways.

Hence, the total number of words $= {}^6C_4 \times 4! = 360$.

EXAMPLE 3 *How many words with or without meaning, can be formed using all the letters of the word EQUATION at a time so that vowels and consonants occur together?* [NCERT]

SOLUTION There are 5 vowels and 3 consonants in the word EQUATION. All vowels can be put together in 5! ways and all consonants can be put together in 3! ways. Considering all vowels as one letter and all consonants as at letter, vowels and consonants can be arranged in 2! ways. Therefore, vowels and consonants can be put together in $5! \times 3! \times 2!$ ways i.e. 1440 ways.

EXAMPLE 4 *How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together ?*

SOLUTION There are 5 vowels and 3 consonants in the word 'EQUATION'. Three vowels out of 5 and 2 consonants out of 3 can be chosen in ${}^5C_3 \times {}^3C_2$ ways. So, there are ${}^5C_3 \times {}^3C_2$ groups each containing 3 consonants and two vowels. Now, each group contains 5 letters which are to be arranged in such a way that 2 consonants occur together. Considering 2 consonants as one letter, we have 4 letters which can be arranged in 4! ways. But two consonants can be put together in 2! ways. Therefore, 5 letters in each group can be arranged in $4! \times 2!$ ways.

Hence, the required number of words $= ({}^5C_3 \times {}^3C_2) \times 4! \times 2! = 1440$.

EXAMPLE 5 *How many words with or without meaning, each 2 of vowels and 3 consonants can be formed from the letters of the word DAUGHTER?* [NCERT]

SOLUTION There are 3 vowels and 5 consonants in the word DAUGHTER out of which 2 vowels and 3 consonants can be chosen in ${}^3C_2 \times {}^5C_3$ ways. These selected five letters can now be arranged in 5! ways.

Hence, required number of words $= {}^3C_2 \times {}^5C_3 \times 5!$

$$= 3 \times 10 \times 120 = 3600$$

EXAMPLE 6 *The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?* [NCERT]

SOLUTION Out of 5 vowels and 21 consonants, 2 vowels and 2 consonants can be chosen in ${}^5C_2 \times {}^{21}C_2$ ways. These selected 4 letters can now be arranged in 4! ways. Therefore, by the fundamental principle of counting, required number of words is ${}^5C_2 \times {}^{21}C_2 \times 4! = 10 \times 210 \times 24 = 50400$.

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